1. Notes on mollifiers

Our objective is to evaluate

$$H(f,P,V,Q) := Q\big(-\frac{1}{L}\frac{d}{d\alpha}\big)Q\big(-\frac{1}{L}\frac{d}{d\beta}\big)g_{f,P,V}(\alpha,\beta)\Big|_{\substack{\alpha=0\\ \beta=0}}$$

where

$$\alpha = \frac{a}{L} \qquad \beta = \frac{b}{L} \qquad L = \log T;$$

$$g_{f,P,V}(\alpha,\beta) := \frac{\Sigma_{f,P,V}(\beta,\alpha) - e^{-(\alpha+\beta)L}\Sigma_{f,P,V}(-\alpha,-\beta)}{(\alpha+\beta)}$$

where

$$\Sigma_{f,P,V}(\alpha,\beta) := \sum_{h,k\le y} \frac{b_h(f,P,V)b_k(f,P,V)(h,k)^{1+\alpha+\beta}}{h^{1+\alpha}k^{1+\beta}}.$$

We are interested in mollifiers with coefficients

$$b_h(f,P,V) = \mu(h)\left(P\big(\frac{\log y/h}{\log y}\big) + \sum_{p|h} f(\gamma_p)V\big(\frac{\log y/h}{\log y}\big)\right)$$

where

$$\gamma_h := \frac{\log h}{\log y}.$$

Here $y = T^\theta$ so that $\log y = \theta L$; in the end any $\theta < 4/7$ is admissible. Also

$$P(x) = \sum_{m=1}^{M} a_m x^m \qquad V(x) = \sum_{n=1}^{N} b_n x^n \qquad f(x) = \sum_{r=2}^{R} c_r x^r.$$

By Möbius inversion we have the approximate identity

$$\Sigma_{f,P,V}(\alpha,\beta) \sim \sum_{d\le y}\frac{1}{d}\left(\sum_{h\le y/d}\frac{b_{hd}(f,P,V)}{h^{1+\alpha}}\right)\left(\sum_{k\le y/d}\frac{b_{kd}(f,P,V)}{k^{1+\beta}}\right).$$

Now

$$\sum_{m\le y/d}\frac{b_{md}(f,P,V)}{m^{1+\alpha}} = \sum_r \frac{a_r}{\log^r y}\sum_{m\le y/d}\frac{\mu(md)\log^r \frac{y}{md}}{m^{1+\alpha}} + \sum_r \frac{b_r}{\log^r y}\sum_{m\le y/d}\frac{\mu(md)h(md)\log^r \frac{y}{md}}{m^{1+\alpha}}.$$

The first term is

$$\sum_r \frac{a_r}{\log^r y} \sum_{m \le \frac{y}{d}} \frac{\mu(md) \log^r \frac{y}{md}}{m^{1+\alpha}} = \sum_r \frac{r! a_r}{\log^r y} \frac{1}{2\pi i} \int_{(2)} \sum_m \frac{\mu(md)}{m^{s+1+\alpha}} \frac{(y/d)^s}{s^{r+1}}\, ds$$

$$\sim \mu(d) \sum_r \frac{r! a_r}{\log^r y} \frac{1}{2\pi i} \int_{(2)} \frac{(s+\alpha)(y/d)^s}{s^{r+1}}\, ds$$

$$\sim \mu(d) \left(\alpha P(\frac{\log y/d}{\log y}) + \frac{P'(\frac{\log y/d}{\log y})}{\log y} \right).$$

We use the notation $x = \frac{y}{d}$ and further $x = y^u$ to rewrite this as

$$\frac{\mu(d)}{\log y}(a\theta P(u) + P'(u))$$

so that

$$\sum_{h \le y/d} \frac{\mu(md) P(\gamma_{md})}{h^{1+\alpha}} = \frac{\mu(d)}{\log y} \frac{d}{d\xi} e^{a\theta\xi} P(u+\xi)\Big|_{\xi=0}.$$

Next, we evaluate

$$\sum_{m \le \frac{y}{d}} \frac{\mu(md)h(md)V\left(\frac{\log \frac{y}{md}}{\log y}\right)}{m^{1+\alpha}} \sim \mu(d)h(d) \sum_{m \le \frac{y}{d}} \frac{\mu(m)V\left(\frac{\log \frac{y}{md}}{\log y}\right)}{m^{1+\alpha}} + \mu(d) \sum_{m \le \frac{y}{d}} \frac{\mu(m)h(m)V\left(\frac{\log \frac{y}{md}}{\log y}\right)}{m^{1+\alpha}}.$$

The first term is

$$\frac{\mu(d)h(d)}{\log y} \frac{d}{d\xi} e^{a\theta\xi} V(u+\xi)\Big|_{\xi=0}.$$

The second term is

$$-\mu(d) \sum_{p \le \frac{y}{d}} \frac{f\left(\frac{\log p}{\log y}\right)}{p^{1+\alpha}} \sum_{m \le \frac{y}{pd}} \frac{\mu(m)V\left(\frac{\log \frac{y}{pmd}}{\log y}\right)}{m^{1+\alpha}}$$

$$\sim -\frac{\mu(d)}{\log y} \sum_{p \le \frac{y}{d}} \frac{f\left(\frac{\log p}{\log y}\right)}{p^{1+\alpha}} \frac{d}{d\xi} e^{a\theta\xi} V\left(\frac{\log \frac{y}{pd}}{\log y} + \xi\right)\Big|_{\xi=0}$$

$$\sim -\frac{\mu(d)}{\log y} \frac{d}{d\xi} e^{a\theta\xi} \int_0^u e^{-a\theta v} f(v) V(u-v+\xi) \frac{dv}{v}\Big|_{\xi=0}.$$

Proposition 1. *using the notation* $u = \frac{\log \frac{y}{d}}{\log y}$ *we have*

$$\sum_{h \le y/d} \frac{b_{hd}(f,P,V)}{h^{1+\alpha}} \sim \frac{\mu(d)}{\log y} \frac{d}{d\xi} e^{a\theta\xi} \left(P(u+\xi) + h(d)V(u+\xi) - \int_0^u V(u-v+\xi) f(v) e^{-a\theta v} \frac{dv}{v} \right)\Big|_{\xi=0}$$

The right side here can also be written as $\mu(d)/\log y$ times

$$a\theta(P(u)+h(d)V(u))+P'(u)+h(d)V'(u)-V(u)f_1(0)-\int_0^u e^{-a\theta v}V(u-v)f_1'(v)\ dv$$

where $f_1(v)=\frac{f(v)}{v}$. We may assume that $f_1(0)=0$; this is then

$$\frac{\mu(d)}{\log y}\left(\frac{d}{d\xi}e^{a\theta\xi}\big(P(u+\xi)+h(d)V(u+\xi)\big)\big|_{\xi=0}-\int_0^u V(u-v)f_1'(v)e^{-a\theta v}\ dv\right)$$

Proposition 2. *We have* $\Sigma_{f,P,V}(\alpha,\beta)$

$$\begin{aligned}= \quad & \frac{1}{\log^2 y}\sum_{d\le y}\frac{\mu^2(d)}{d}\left(\frac{d}{d\xi}e^{a\theta\xi}\big(P(u+\xi)+h(d)V(u+\xi)\big)\big|_{\xi=0}-\int_0^u V(u-v)f_1'(v)e^{-a\theta v}\ dv\right)\\ & \times\left(\frac{d}{d\eta}e^{b\theta\eta}\big(P(u+\eta)+h(d)V(u+\eta)\big)\big|_{\eta=0}-\int_0^u V(u-w)f_1'(w)e^{-b\theta w}\ dw\right)\end{aligned}$$

We have a lemma to help with the summation over d.

Lemma 1. *Let* $\gamma=c/L$*;*

$$F_1(u)=\int_0^u f_1(v)\ dv \qquad F_2(u)=\int_0^u vf_1(v)^2\ dv \qquad F_3(u)=\int_0^u (f_1\star f_1)(v)\ dv$$

and

$$h(d)=\sum_{p|d}f\left(\frac{\log p}{\log y}\right).$$

Also, for any function G, let $(T_cG)(u)=e^{-c\theta u}G(u)$. We have

$$\frac{1}{\log y}\sum_{d\le y}\frac{\mathcal{P}\left(\frac{\log y/d}{\log y}\right)}{d^{1+\gamma}}\sim\int_0^1 e^{-c\theta(1-u)}\mathcal{P}(u)\ du=(\mathcal{P}\star T_c1)(1);$$

$$\frac{1}{\log y}\sum_{d\le y}\frac{h(d)\mathcal{P}\left(\frac{\log y/d}{\log y}\right)}{d^{1+\gamma}}\sim\int_0^1 e^{-c\theta(1-u)}\mathcal{P}(u)\left(\int_0^{1-u}f_1(v)\ dv\right)\ du=(\mathcal{P}\star T_cF_1)(1).$$

Finally

$$\begin{aligned}\frac{1}{\log y}\sum_{d\le y}\frac{h^2(d)\mathcal{P}\left(\frac{\log y/d}{\log y}\right)}{d} \quad &\sim\quad \int_0^1\mathcal{P}(u)\left(\int_0^{1-u}vf_1^2(v)\ dv+\iint_{v_1+v_2\le 1-u}f_1(v_1)f_1(v_2)\ dv_1\ dv_2\right)\ du\\ &=\quad (\mathcal{P}\star(F_2+F_3))(1).\end{aligned}$$

We multiply out the sum over d and apply the lemma. Let's first consider

$$\frac{1}{\log y}\sum_{d\le y}\frac{\mu^2(d)}{d}\big(P(u+\xi)+h(d)V(u+\xi)\big)\big(P(u+\eta)+h(d)V(u+\eta)\big)$$

By the lemma, this is asymptotic to

$$\begin{aligned}\Sigma_1(\xi,\eta) &:= \int_0^1 P(u+\xi)P(u+\eta)\ du \\ &+ \int_0^1 \big(P(u+\xi)V(u+\eta)+V(u+\xi)P(u+\eta)\big)\int_0^{1-u} f_1(v)\ dv\ du \\ &+ \int_0^1 V(u+\xi)V(u+\eta)\left(\int_0^{1-u} v f_1^2(v)\ dv + \iint_{v_1+v_2\le 1-u} f_1(v_1)f_1(v_2)\ dv_1\ dv_2\right)\ du\end{aligned}$$

Next let's consider the cross terms

$$\frac{1}{\log y}\sum_{d\le y}\frac{\mu^2(d)}{d}\big(P(u+\xi)+h(d)V(u+\xi)\big)\int_0^u V(u-w)f_1'(w)e^{-b\theta w}\ dw$$

and

$$\frac{1}{\log y}\sum_{d\le y}\frac{\mu^2(d)}{d}\int_0^u V(u-v)f_1'(v)e^{-a\theta v}\ dv\big(P(u+\eta)+h(d)V(u+\eta)\big)$$

The first is asymptotic to $\Sigma_2(b,\xi)$ and the second to $\Sigma_2(a,\eta)$ where

$$\Sigma_2(a,\eta) := \int_0^1\int_0^u V(u-v)f_1'(v)e^{-a\theta v}\ dv\left(P(u+\eta)+V(u+\eta)\int_0^{1-u} f_1(w)\ dw\right)\ du$$

Finally we have

$$\frac{1}{\log y}\sum_{d\le y}\frac{\mu^2(d)}{d}\int_0^u V(u-v)f_1'(v)e^{-a\theta v}\ dv\int_0^u V(u-w)f_1'(w)e^{-b\theta w}\ dw$$

By the lemma this is asymptotic to

$$\Sigma_3(a,b) := \int_0^1\int_0^u V(u-v)f_1'(v)e^{-a\theta v}\ dv\int_0^u V(u-w)f_1'(w)e^{-b\theta w}\ dw\ du$$

The following lemma is useful for the next step.

Lemma 2.

$$\frac{e^{b\xi+a\eta}-e^{-(a+b)}e^{-a\xi-b\eta}}{a+b} = e^{-a\xi-b\eta}\int_0^1 e^{-(a+b)x}\ dx + (\xi+\eta)\int_0^1 e^{a(\eta(1-x)-\xi x)+b(\xi(1-x)-\eta x)}\ dx$$

Corollary 1.

$$\begin{aligned}&Q\left(-\frac{d}{da}\right)Q\left(-\frac{d}{db}\right)\frac{e^{b\xi+a\eta}-e^{-(a+b)}e^{-a\xi-b\eta}}{a+b}\Bigg|_{\substack{a=0\\b=0}} \\ &\qquad = \int_0^1 Q(x+\xi)Q(x+\eta)\ dx + (\xi+\eta)\int_0^1 Q(\xi x-\eta(1-x))Q(\eta x-\xi(1-x))\ dx\end{aligned}$$

The contribution from the terms of Σ_1 have the shape $\int_0^1 G(u)R(u+\xi)S(u+\eta)\,du$ (applying the above with ξ and η replaced by $\theta\xi$ and $\theta\eta$); they give

$$Q\left(-\frac{d}{da}\right)Q\left(-\frac{d}{db}\right)\frac{1}{\theta}\frac{d}{d\xi}\frac{d}{d\eta}\int_0^1 G(u)R(u+\xi)S(u+\eta)\,du\frac{e^{b\theta\xi+a\theta\eta}-e^{-a-b}e^{-a\theta\xi-b\theta\eta}}{a+b}\Bigg|_{\substack{\xi=\eta=0\\a=b=0}}$$

$$=\frac{1}{\theta}\frac{d}{d\xi}\frac{d}{d\eta}\int_0^1 G(u)R(u+\xi)S(u+\eta)\,du\Bigg(\int_0^1 Q(x+\theta\xi)Q(x+\theta\eta)\,dx$$
$$+\theta(\xi+\eta)\int_0^1 Q(\theta(\xi x-\eta(1-x)))Q(\theta(\eta x-\xi(1-x)))\,dx\Bigg)\Bigg|_{\xi=\eta=0}$$

which simplifies to

$$\theta\int_0^1 R(u)S(u)G(u)\,du\int_0^1 Q'(x)^2\,dx+\frac{1}{\theta}\int_0^1 R'(u)S'(u)G(u)\,du\int_0^1 Q(x)^2\,dx$$
$$+\frac{Q(0)^2+Q(1)^2}{2}\int_0^1\big(R(u)S'(u)+R'(u)S(u)\big)G(u)\,du$$

In the case that $R=S=P$ and $G=1$ we get

$$\theta\int_0^1 P(u)^2\,du\int_0^1 Q'(x)^2\,dx+\frac{1}{\theta}\int_0^1 P'(u)^2\,du\int_0^1 Q(x)^2\,dx$$
$$+\frac{Q(0)^2+Q(1)^2}{2}P(1)^2.$$

In the case that $R=P$, $S=V$ and $G(u)=F_1(1-u)$ we get

$$\theta\int_0^1 P(u)V(u)F_1(1-u)\,du\int_0^1 Q'(x)^2\,dx+\frac{1}{\theta}\int_0^1 P'(u)V'(u)F_1(1-u)\,du\int_0^1 Q(x)^2\,dx$$
$$+\frac{Q(0)^2+Q(1)^2}{2}\int_0^1 (PV)'(u)F_1(1-u)\,du.$$

There are two copies of this expression. In the case that $R=S=V$ and $G(u)=F_2(1-u)+F_3(1-u)$ we get

$$\theta\int_0^1 V(u)^2(F_2(1-u)+F_3(1-u))\,du\int_0^1 Q'(x)^2\,dx$$
$$+\frac{1}{\theta}\int_0^1 V'(u)^2(F_2(1-u)+F_3(1-u))\,du\int_0^1 Q(x)^2\,dx$$
$$+(Q(0)^2+Q(1)^2)\int_0^1 V(u)V'(u)(F_2(1-u)+F_3(1-u))\,du$$

We turn to Σ_2 and calculate (using the corollary with $\xi = -\theta v$ and η replaced by $\theta\eta$) to get

$$\frac{1}{\theta}Q\left(-\frac{d}{da}\right)Q\left(-\frac{d}{db}\right)\frac{\frac{d}{d\eta}e^{b\theta\eta}\Sigma_2(a,\eta)-e^{-a-b}\frac{d}{d\eta}e^{a\theta\eta}\Sigma_2(b,\eta)}{a+b}\Bigg|_{\substack{a=b=0\\ \eta=0}}$$

$$=\frac{1}{\theta}Q\left(-\frac{d}{da}\right)Q\left(-\frac{d}{db}\right)\frac{d}{d\eta}\int_0^1\int_0^u V(u-v)f_1'(v)(P(u+\eta)+V(u+\eta)F_1(1-u))$$

$$\times\frac{e^{a\theta\eta-b\theta v}-e^{-a-b}e^{-b\theta\eta+a\theta v}}{a+b}\;dv\;du\Bigg|_{\substack{a=b=0\\ \eta=0}}$$

$$=\frac{1}{\theta}\frac{d}{d\eta}\int_0^1\int_0^1\int_0^u V(u-v)f_1'(v)(P(u+\eta)+V(u+\eta)F_1(1-u))$$

$$\times\Big(Q(x-\theta v)Q(x+\theta\eta)+\theta(\eta-v)Q(-\theta vx-\theta\eta(1-x))Q(\theta\eta x+\theta v(1-x))\Big)\;dx\Bigg|_{\eta=0}$$

where

$$F_1(u)=\int_0^u f_1(v)\;dv.$$

The above evaluates to

$$\frac{1}{\theta}\int_0^1\int_0^1(P'(u)+F_1(1-u)V'(u))\int_0^u V(u-v)f_1'(v)Q(x)Q(x-\theta v)\;dv\;dx\;du$$

$$-\int_0^1\int_0^1(P(u)+F_1(1-u)V(u))\int_0^u V(u-v)vf_1'(v)xQ(-\theta vx)Q'(\theta v(1-x))\;dv\;du\;dx$$

$$-\int_0^1\int_0^1(P(u)+F_1(1-u)V(u))\int_0^u V(u-v)vf_1'(v)(1-x)Q'(-\theta vx)Q(\theta v(1-x))\;dv\;du\;dx$$

$$+\int_0^1\int_0^1\int_0^u\Big(\big(P(u)+V(u)F_1(1-u)\big)f_1'(v)V(u-v)\big(Q(\theta v(1-x))Q(-\theta vx)+Q(x-\theta v)Q'(x)\big)$$

$$+\big(P'(u)+V'(u)F_1(1-u)\big)f_1'(v)V(u-v)Q(\theta v(1-x))Q(-\theta vx)\Big)\;dv\;du\;dx$$

There are two copies of this expression.

If we apply the corollary (with $\xi=-\theta v,\eta=-\theta w$) to Σ_3 we have

$$\frac{1}{\theta}Q\left(-\frac{d}{da}\right)Q\left(-\frac{d}{db}\right)\frac{\Sigma_3(b,a)-e^{-a-b}\Sigma_3(-a,-b)}{a+b}\Bigg|_{a=b=0}$$

$$=\frac{1}{\theta}\int_0^1\int_0^1\int_0^u\int_0^u V(u-v)f_1'(v)V(u-w)f_1'(w)\Big(Q(x-\theta v)Q(x-\theta w)$$

$$-\theta(v+w)Q(-\theta vx+\theta w(1-x))Q(-\theta wx+\theta v(1-x))\Big)\;dv\;dw\;du\;dx$$

2. Adding it all up

We get

$$
\begin{aligned}
&\theta \int_0^1 P(u)^2 \, du \int_0^1 Q'(x)^2 \, dx + \frac{1}{\theta} \int_0^1 P'(u)^2 \, du \int_0^1 Q(x)^2 \, dx + \frac{Q(0)^2 + Q(1)^2}{2} P(1)^2 \\
&\quad + 2\theta \int_0^1 P(u)V(u)F_1(1-u) \, du \int_0^1 Q'(x)^2 \, dx + \frac{2}{\theta} \int_0^1 P'(u)V'(u)F_1(1-u) \, du \int_0^1 Q(x)^2 \, dx \\
&\quad + 2\frac{Q(0)^2 + Q(1)^2}{2} \int_0^1 (PV)'(u)F_1(1-u) \, du \\
&\quad + \theta \int_0^1 V(u)^2 (F_2(u) + F_3(u)) \, du \int_0^1 Q'(x)^2 \, dx + \frac{1}{\theta} \int_0^1 V'(u)^2 (F_2(u) + F_3(u)) \, du \int_0^1 Q(x)^2 \, dx \\
&\quad + \frac{Q(0)^2 + Q(1)^2}{2} \int_0^1 2V(u)V'(u)(F_2(u) + F_3(u)) \, du \\
&\quad + \frac{2}{\theta} \int_0^1 \int_0^1 (P'(u) + F_1(1-u)V'(u)) \int_0^u V(u-v) f_1'(v) Q(x) Q(x - \theta v) \, dv \, dx \, du \\
&\quad + 2 \int_0^1 \int_0^1 (P(u) + F_1(1-u)V(u)) \int_0^u V(u-v) f_1'(v) \frac{d}{dv} Q(-\theta v x) Q(\theta v(1-x)) \, dv \, du \, dx \\
&\quad + 2 \int_0^1 \int_0^1 \int_0^u \Big(\big(P(u) + V(u)F_1(1-u)\big) f_1'(v) V(u-v) \big(Q(\theta v(1-x)) Q(-\theta v x) + Q(x - \theta v) Q'(x) \big) \\
&\qquad + 2\big(P'(u) + V'(u)F_1(1-u)\big) f_1'(v) V(u-v) Q(\theta v(1-x)) Q(-\theta v x) \Big) \, dv \, du \, dx \\
&\quad + \frac{1}{\theta} \int_0^1 \int_0^1 \int_0^u \int_0^u V(u-v) f_1'(v) V(u-w) f_1'(w) \Big(Q(x - \theta v) Q(x - \theta w) \\
&\qquad - \theta(v+w) Q(-\theta v x + \theta w(1-x)) Q(-\theta w x + \theta v(1-x)) \Big) \, dv \, dw \, du \, dx
\end{aligned}
$$

Let

$$W(u) = P(u) + V(u)F_1(1-u)$$

and

$$E(u) = F_2(1-u) + F_3(1-u) - F_1(1-u)^2.$$

Then we can write the above as

$$
\begin{aligned}
&\theta \int_0^1 (W(u)^2 - E(u))\ du \int_0^1 Q'(x)^2\ dx \\
&\quad + \frac{1}{\theta} \int_0^1 (W'(u)^2 - E(u))\ du \int_0^1 Q(x)^2\ dx + \frac{Q(0)^2 + Q(1)^2}{2} P(1)^2 \\
&\quad + (Q(0)^2 + Q(1)^2) \int_0^1 P(u)V(u) f_1(1-u)\ du \\
&\quad + (Q(0)^2 + Q(1)^2) \int_0^1 V(u)^2((1-u)f_1(1-u)^2 + (f_1 \star f_1)(1-u))\ du \\
&\quad + \frac{2}{\theta} \int_0^1 \int_0^1 (W'(u) + V(u)f_1(1-u)) \int_0^u V(u-v) f_1'(v) Q(x) Q(x - \theta v)\ dv\ dx\ du \\
&\quad + 2 \int_0^1 \int_0^1 \int_0^u W(u)V(u-v) f_1'(v) \frac{d}{dv} Q(-\theta v x) Q(\theta v(1-x))\ dv\ du\ dx \\
&\quad + 2 \int_0^1 \int_0^1 \int_0^u \Big(W(u)V(u-v) f_1'(v) \big(Q(-\theta v x) Q(\theta v(1-x)) + Q(x - \theta v) Q'(x) \big) \\
&\qquad\quad + 2(W'(u) + V(u) f_1(1-u)) f_1'(v) V(u-v) Q(\theta v(1-x)) Q(-\theta v x) \Big)\ dv\ du\ dx \\
&\quad + \frac{1}{\theta} \int_0^1 \int_0^1 \int_0^u \int_0^u V(u-v) f_1'(v) V(u-w) f_1'(w) \Big(Q(x - \theta v) Q(x - \theta w) \\
&\qquad\quad - \theta(v+w) Q(-\theta v x + \theta w(1-x)) Q(-\theta w x + \theta v(1-x)) \Big)\ dv\ dw\ du\ dx
\end{aligned}
$$

3. Proof of Lemma 2

Proof. First note that

$$\frac{e^{b\xi + a\eta} - e^{-a\xi - b\eta}}{a+b} = (\xi + \eta) \int_0^1 e^{a(\eta(1-x) - \xi x) + b(\xi(1-x) - \eta x)}\ dx$$

This can be rewritten as

$$e^{b\xi + a\eta} = e^{-a\xi - b\eta} + (a+b)(\xi + \eta) \int_0^1 e^{a(\eta(1-x) - \xi x) + b(\xi(1-x) - \eta x)}\ dx$$

so that

$$
\begin{aligned}
\frac{e^{b\xi + a\eta} - e^{-(a+b)} e^{-a\xi - b\eta}}{a+b} &= \frac{e^{-a\xi - b\eta}(1 - e^{-(a+b)})}{a+b} \\
&\quad + (\xi + \eta) \int_0^1 e^{a(\eta(1-x) - \xi x) + b(\xi(1-x) - \eta x)}\ dx
\end{aligned}
$$

which can be rewritten as

$$e^{-a\xi-b\eta}\int_0^1 e^{-(a+b)x}\ dx+(\xi+\eta)\int_0^1 e^{a(\eta(1-x)-\xi x)+b(\xi(1-x)-\eta x)}\ dx$$

□