AVERAGES OF LONG DIRICHLET POLYNOMIALS

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1. Introduction and statement of results

It has been conjectured by Keating and Snaith that

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2k}\, dt \sim \frac{g_k a_k (\log T)^{k^2}}{k^2!}$$

where for positive integer k,

$$a_k = \prod_p \left(1-\frac{1}{p}\right)^{(k-1)^2} \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}^2}{p^j}$$

and

$$g_k = \frac{k^2!}{1^1\cdot 2^2\cdot \ldots k^k\cdot (k+1)^{k-1}\cdot \cdots \cdot (2k-1)^1}.$$

This has been proven for $k=1$ and $k=2$. The method of proof involves approximating $\zeta(s)$ or $\zeta(s)^2$ by appropriate Dirichlet polynomials and analyzing the mean-square of such. In the pursuit of proving the above conjecture for values of k larger than 2, it may be of some interest to consider in general the mean square of Dirichlet polynomials with coefficients $d_k(n)$ where

$$\zeta(s)^k = \sum_{n=1}^\infty \frac{d_k(n)}{n^s}.$$

Thus, we consider

$$I_k(T,N) = \int_0^T \sum_{n=1}^N \left|\frac{d_k(n)}{n^{1/2+it}}\right|^2\, dt$$

for various values of k and N.

Here we present a method which will lead to conjectural values for

$$M_k(\alpha) = \lim_{T\to\infty} \frac{(k^2)!}{a_k T(\log T)^{k^2}} I_k(T,N)$$

for integer values of k and $N = T^\alpha$ with $\alpha > 0$. In particular we are interested in unit intervals of α between 0 and k. For example it can be shown that

$$\frac{1}{T}\int_0^T \left|\sum_{n\le N} \frac{1}{n^{1/2+it}}\right|^2\, dt \sim \begin{cases} \log N & \text{if } N\le T\\ \log T & \text{if } N > T\end{cases}$$

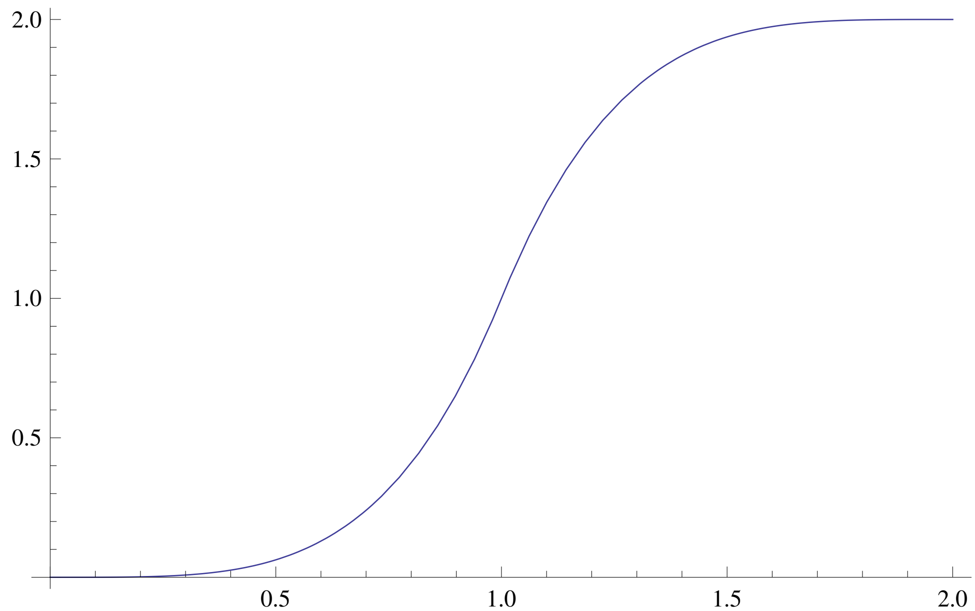

FIGURE 1. The plot of $M_2(\alpha)$ for $0 < \alpha < 2$.

This translates to

$$M_1(\alpha) = \begin{cases} \alpha & \text{if } 0 \le \alpha \le 1 \\ 1 & \text{if } 1 < \alpha \end{cases}$$

Also, it can likely be proven that

$$M_2(\alpha) = \begin{cases} \alpha^4 & \text{if } 0 \le \alpha \le 1 \\ -\alpha^4 + 8\alpha^3 - 24\alpha^2 + 32\alpha - 14 & \text{if } 1 < \alpha \le 2 \\ 2 & \text{if } 2 < \alpha \end{cases}$$

Next, we conjecture that

$$M_3(\alpha) = \begin{cases} \alpha^9 & \text{if } 0 \le \alpha \le 1 \\ -2\alpha^9 + 27\alpha^8 - 324\alpha^7 + 2268\alpha^6 - 8694\alpha^5 + & \\ \quad 19278\alpha^4 - 25452\alpha^3 + 19764\alpha^2 - 8343\alpha + 1479 & \text{if } 1 < \alpha \le 2 \\ \alpha^9 - 27\alpha^8 + 324\alpha^7 - 2268\alpha^6 + 10206\alpha^5 & \\ \quad -30618\alpha^4 + 61236\alpha^3 - 78732\alpha^2 + 59049\alpha - 19641 & \text{if } 2 \le \alpha \le 3 \\ 42 & \text{if } 3 \le \alpha \end{cases}$$

This is a consequence of the conjecture of [CFKRS] known as "the recipe." We will sketch its derivation later.

The polynomials here are interesting because of their smoothness properties. The graphs of $M_2(\alpha)$ and $M_3(\alpha)$ are included. Notice that they are very smooth, monotonic, and are symmetric.

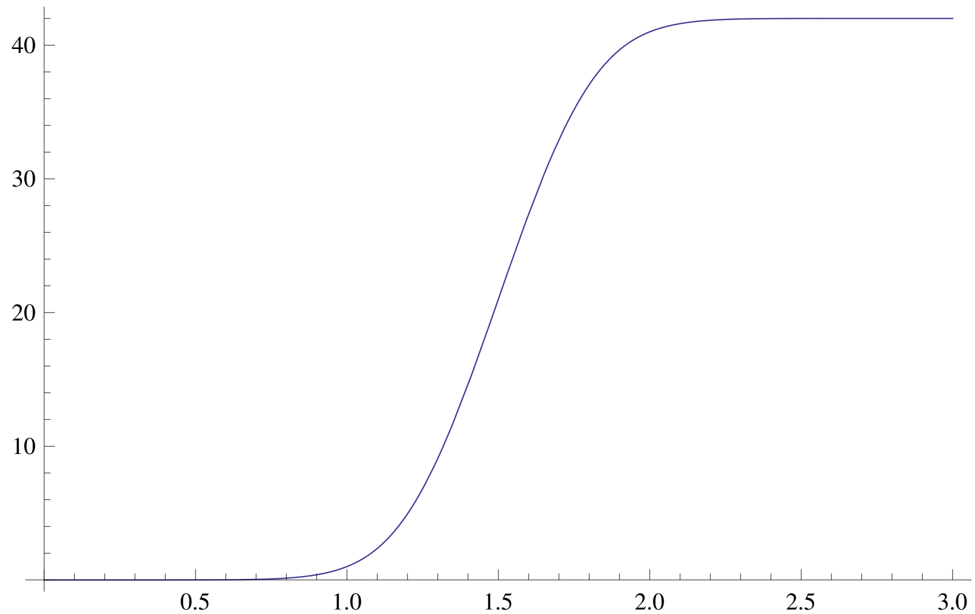

FIGURE 2. The plot of $M_3(\alpha)$ for $0 < \alpha < 3$.

In fact, $M_3(\alpha)$ is 9-times continuously differentiable at $\alpha = 0$ and $\alpha = 3$ and is 5-times differentiable at $\alpha = 1$ and $\alpha = 2$. It can be proven that the only piecewise polynomial $f(\alpha)$ (with pieces of degree at most 9) which is 0 for $\alpha < 0$ is 42 for $\alpha \geq 3$, is monotonic, and satisfies $f(3 - \alpha) = 42 - f(\alpha)$ and has the same smoothness properties as $M_3(\alpha)$ is $f(\alpha) = M_3(\alpha)$. Note that the symmetry together with $M_3(\alpha) = \alpha^9$ for $0 < \alpha < 1$ implies that for $2 < \alpha < 3$ we have

$$M_3(\alpha) = (\alpha - 3)^9 + 42.$$

which only leaves the range $1 < \alpha < 2$ in question. Let $P(\alpha)$ be the polynomial that agrees with $M_3(\alpha)$ in the range $1 < \alpha < 2$. Then it satisfies $P(\alpha) + P(3 - \alpha) = 42$; this determines half of its 10 coefficients. Then the 5 times smoothness at $\alpha = 1$ determine the other 5.

2. A PROOF OF THE $k = 2$ CASE

We sketch a possible proof of the $k = 2$ case. First of all, with $s = 1/2 + it$ and $\alpha, \beta, \gamma, \delta \ll (\log T)^{-1}$ it is a theorem (but whose proof is not written down in full details anywhere) that

$$(1)\int_0^T \zeta(s+\alpha)\zeta(s+\beta)\zeta(1-s+\gamma)\zeta(1-s+\delta)\; dt = \int_0^T \mathcal{Z}_t(\alpha,\beta,\gamma,\delta)\; dt + O(T^{2/3+\epsilon}),$$

where

$$\begin{aligned}\mathcal{Z}_t(\alpha,\beta,\gamma,\delta) &= Z(\alpha,\beta,\gamma,\delta) + \left(\frac{t}{2\pi}\right)^{-\alpha-\gamma} Z(-\gamma,\beta,-\alpha,\delta) + \left(\frac{t}{2\pi}\right)^{-\alpha-\delta} Z(-\delta,\beta,\gamma,-\alpha)\\ &\quad + \left(\frac{t}{2\pi}\right)^{-\beta-\gamma} Z(\alpha,-\gamma,-\beta,\delta) + \left(\frac{t}{2\pi}\right)^{-\beta-\delta} Z(\alpha,-\delta,\gamma,-\beta)\\ &\quad + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta} Z(-\gamma,-\delta,-\alpha,-\beta)\end{aligned}$$

where

$$Z(\alpha,\beta,\gamma,\delta) = \frac{\zeta(1+\alpha+\gamma)\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)\zeta(1+\delta)}{\zeta(2+\alpha+\beta+\gamma+\delta)}$$

This theorem, possibly with a weaker error term, could be proven in the case that real parts of the $\alpha,\beta,\gamma,\delta$ are small but that the imaginary parts can be as large as T; (Sandro Bettin did this for the mean square case, see [B]). We will assume this uniform version of the fourth moment. By Perron's formula we have

$$\begin{aligned}I_2(T,N) &= \int_0^T \frac{1}{(2\pi i)^2}\iint_{z,w} \zeta(s+w)^2\zeta(1-s+z)^2 \frac{N^w}{w}\frac{N^z}{z}\,dw\,dz\,dt\\ &= \frac{1}{(2\pi i)^2}\iint_{z,w}\frac{N^{w+z}}{wz}\int_0^T \zeta(s+w)^2\zeta(1-s+z)^2\,dt\,dw\,dz\end{aligned}$$

We evaluate the inner integral over t using a limiting case of (1) with $\alpha = \beta = w$ and $\gamma = \delta = z$. Also, we are only interested in the leading order term, so, for example, the denominator in the recipe formula above just becomes $\zeta(2) = \pi^2/6$ and we replace $\zeta(1+x)$ by $1/x$, $(t/2\pi)^{-\alpha}$ by $T^{-\alpha}$, etc. In this context then, we have

$$\frac{\zeta(2)^{-1}}{T}\int_0^T \zeta(s+w)^2\zeta(1-s+z)^2\,dt \sim \frac{1-(2+(w+z)^2)T^{-w-z}\log^2 T - 2T^{-2w-2z}}{(w+z)^4}$$

Inserting this above we find that

$$I_2(T,N) \sim \frac{\zeta(2)^{-1}T}{(2\pi i)^2}\iint_{z,w}\frac{N^{w+z}}{wz}\frac{(1-(2+(w+z)^2)T^{-w-z}\log^2 T - 2T^{-2w-2z})}{(w+z)^4}\,dw\,dz.$$

The integrals over z and w are for the real parts of z and w being small but positive. We can see from this formula that we will get different answers when $N<T$, $T<N<T^2$, and $T^2<N$. For example, if $T<N<T^2$ we will move the paths of integration to the right (and so get 0) for the terms which involve T^{-2w-2z}. If $N<T$ then we do likewise for the terms which involve T^{-z-w} or T^{-2w-2z}. For the rest of the terms we move the paths to the

left and collect the residues at $w = 0$ and $z = 0$. In this way we find that $I_2(T,N) \sim \frac{T}{4!\zeta(2)}\times$

$$\times \begin{cases} \log^4 N & \text{if } N < T \\ 8\log^3 N \log T + 32 \log N \log^3 T - 24 \log^2 N \log^2 T - \log^4 N - 14 \log^4 T & \\ & \text{if } T < N < T^2 \\ 2\log^4 T & \text{if } T^2 < N \end{cases}$$

The result about $M_2(\alpha)$ follows.

3. Derivation of the case $k = 3$

We use the conjecture of [CFKRS]. Let

$$Z_\zeta(A;B) = \prod_{\alpha\in A, \beta\in B} \zeta(1+\alpha+\beta)$$

and

$$\mathcal{A}(A;B) = \prod_p \prod_{\alpha\in A,\beta\in B} \left(1 - \frac{1}{p^{1+\alpha+\beta}}\right) \times \int_0^1 \prod_{\alpha\in A} z_{p,\theta}(1/2+\alpha) \prod_{\beta\in B} z_{p,-\theta}(1/2+\beta)\ d\theta$$

where $z_{p,\theta}(x) = 1/(1-e(\theta)/p^x)$. Then

$$\int_0^T \prod_{\alpha\in A} \zeta(1/2+i\tau+\alpha) \prod_{\beta\in B} \zeta(1/2-i\tau+\beta)\ d\tau = \int_0^T \sum_{\substack{S\subset A \\ T\subset B \\ |S|=|T|}} e^{-\ell(\sum s + \sum t)} \mathcal{A} Z_\zeta(\overline{S}\cup(-T);\overline{T}\cup(-S))\ d\tau + O(T^{1/2+\epsilon}).$$

where $\ell = \log \frac{t}{2\pi}$.

We use the above with A and B being sets of cardinality 3. A limiting argument that allows for A and B to be multisets $A = \{w,w,w\}$ and $B = \{z,z,z\}$ implies that

$$\frac{1}{T}\int_0^T \zeta(s+w)^3 \zeta(1-s+z)^3\ dt \sim F_3(w,z)$$

where

$$F_3(w,z) = \frac{1}{4}(w+z)^{-9}(4 + T^{-w-z}(-w^4\log^4(T) - 4w^3z\log^4(T) + 4w^3\log^3(T)$$
$$-6w^2z^2\log^4(T) + 12w^2z\log^3(T) - 12w^2\log^2(T) - 4wz^3\log^4(T)$$
$$+12wz^2\log^3(T) - 24wz\log^2(T) - z^4\log^4(T) + 4z^3\log^3(T)$$
$$-12z^2\log^2(T) - 12) + T^{-2w-2z}(w^4\log^4(T) + 4w^3z\log^4(T) + 4w^3\log^3(T)$$
$$+6w^2z^2\log^4(T) + 12w^2z\log^3(T) + 12w^2\log^2(T) + 4wz^3\log^4(T)$$
$$+12wz^2\log^3(T) + 24wz\log^2(T) + z^4\log^4(T) + 4z^3\log^3(T)$$
$$+12z^2\log^2(T) + 12) - 4T^{-3w-3z})$$

We compute

$$I_3(T) = \frac{1}{(2\pi i)^2}\iint_{w,z}\frac{N^{w+z}}{wz}F_3(w,z)\,dw\,dz$$

for various ranges of N. If $N < T$ only the first term matters; if $T < N < T^2$ then the terms with T^{-w-z} also contribute; if $T^2 < N < T^3$ then we must also include the terms with T^{-2w-2z}; if $N > T^3$ then we include all of the terms. Computing residues at $w = 0$ and $z = 0$ leads to the above result for M_3.

4. $k = 4$

We know that $M_4(\alpha) = \alpha^{16}$ for $0 < \alpha < 1$. We know also that $M_4(\alpha) = 24024$ for $\alpha \geq 4$ and that $M_4(\alpha) = 24024 - M_4(4-\alpha)$ for all α, so that determines $M_4(\alpha)$ for $3 < \alpha < 4$. One might guess that it will be 9 times differentiable at $\alpha = 1$ and $\alpha = 3$. And 7 times differentiable at $\alpha = 2$.

We can use a result in [CG] to conjecturally determine $M_4(\alpha)$ (and indeed any $M_k(\alpha)$) for $1 < \alpha < 2$. From that paper, which is based on the predicted behavior of divisor correlations

$$\sum_{n\leq x} d_k(n)d_k(n+h)$$

we have

Conjecture 1. *For any positive integer k, we conjecture that $M_k(\alpha)$ exists and that*

$$M_k(\alpha) = \alpha^{k^2}\left(1 - \sum_{n=0}^{k^2-1}(-1)^n\left(1-\alpha^{-n-1}\right)\binom{k^2}{n+1}\gamma_k(n)\right)$$

for $1 < \alpha < 2$ where

$$\gamma_k(n) = \sum_{1\leq i,j\leq k}\binom{k}{i}\binom{k}{j}\binom{n-1}{i-1,j-1,n-i-j+1};$$

also

$$\gamma_k(0) = k.$$

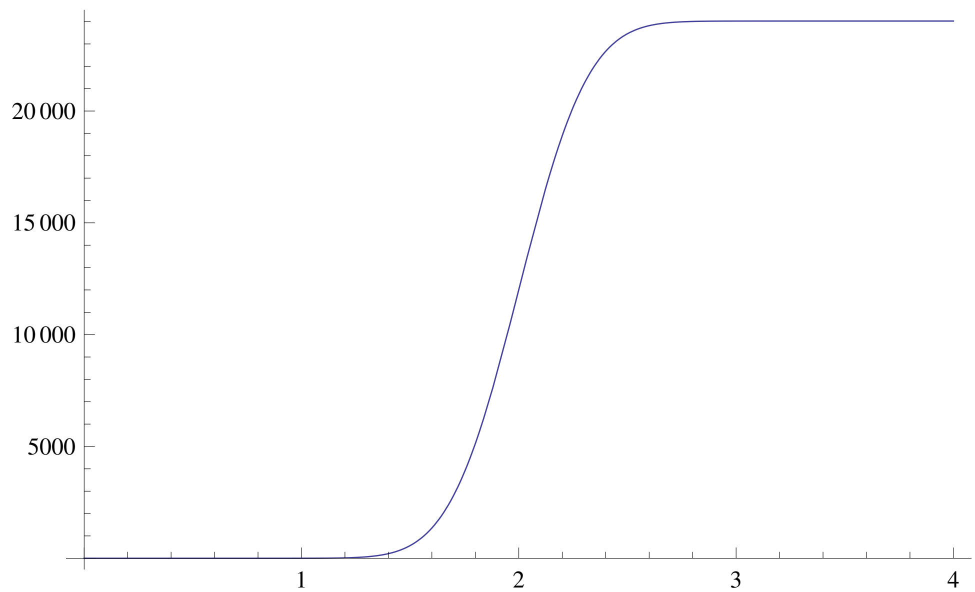

FIGURE 3. The plot of $M_4(\alpha)$ for $0 < \alpha < 4$.

In [CK1-3] we confirm that the correlation method of [CG] agrees with the prediction of the recipe in [CFKRS]. These independent lines of reasoning thus both give

$$\begin{aligned} M_4(\alpha) = {} & -3\alpha^{16} + 64\alpha^{15} - 1920\alpha^{14} + 35840\alpha^{13} - 393120\alpha^{12} + 2725632\alpha^{11} - 12684672\alpha^{10} \\ & +41367040\alpha^9 - 97348680\alpha^8 + 168351040\alpha^7 - 215767552\alpha^6 + 204701952\alpha^5 \\ & -141989120\alpha^4 + 70035840\alpha^3 - 23281920\alpha^2 + 4679424\alpha - 429844 \end{aligned}$$

for $1 < \alpha < 2$, which does satisfy the aforementioned smoothness conditions. With this information we can construct all of $M_4(\alpha)$.

5. Remarks

With a lot more work we could find an explicit formula for $M_k(\alpha)$ for $2 < \alpha < 3$, or indeed for any initial interval, using the recipe method. Also, we have a preliminary version of a new method - the convolution coefficient correlation method - which would give an independent avenue into determining the $M_k(\alpha)$. However, we suspect that there are simple smoothness conditions which would completely characterize $M_k(a)$. We are not sure exactly what these are. However, the following may be a start.

Conjecture 2. *For any positive integer k the function $M_k(\alpha)$ is $(k-1)^2$ times continuously differentiable at $\alpha = 1$.*

We have checked this conjecture for $k \leq 7$ using the proposed formulas above for $M_k(\alpha)$ for $0 < \alpha < 2$.

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