

# WHICH NUMBERS ARE NOT THE SUM PLUS THE PRODUCT OF THREE POSITIVE INTEGERS?

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ABSTRACT. We investigate the number  $R_3(n)$  of representations as the sum plus the product of three positive integers. On average,  $R_3(n)$  is  $\frac{1}{2} \log^2 n$ . We give an upper bound for  $R_3(n)$  and an upper bound for the number of  $n \leq N$  such that  $R_3(n) = 0$ . We conjecture that  $R_3(n) = 0$  infinitely often.

## 1. INTRODUCTION

It is well-known that the number of representations of an integer as a product of  $k$  positive integers, denoted  $\tau_k(n)$ , satisfies  $\tau_k(n) \ll_\epsilon n^\epsilon$  for any  $\epsilon > 0$  as  $n$  goes to infinity (see Section 1.1 for notation). The proof of this standard result relies crucially on the fact that  $\tau_k$  is a multiplicative function. For  $k = 3$  this amounts to counting integral solutions of

$$n = xyz.$$

The slightest perturbation of this problem turns it from an exercise to a seriously hard research question. One such example is the problem of trying to estimate the number of representations of a number  $n$  as the sum plus the product of three positive integers, i.e.

$$(1) \quad n = xyz + x + y + z.$$

Let  $R_3(n)$  denote the number of solutions to (1) in positive integers  $x, y, z$ . Here are the values of  $R_3(n)$  for  $1 \leq n \leq 12$ :

$$0, 0, 0, 1, 0, 3, 0, 3, 3, 3, 0, 9$$

Our motivating problem was the following conjecture:

**Conjecture 1.** *For any  $\epsilon > 0$  there is a  $C(\epsilon) > 0$  such that*

$$R_3(n) \leq C(\epsilon)n^\epsilon$$

*for all positive integers  $n$ .*

While we were unable to prove this conjecture, we were able to find and prove a number of other interesting results as well as formulate some new conjectures.

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**1.1. Notation.** We use some notation that is standard in analytic number theory but perhaps not standard to everybody. In this section, we define this notation for the reader.

Suppose  $f(n)$  and  $g(n)$  are functions defined on the positive integers. Then,  $f(n) = O(g(n))$  means that there exists some  $M > 0$  and  $N_0$  (which could depend on  $M$ ) such that  $|f(n)| \leq Mg(n)$  for all  $n \geq N_0$ . Also,  $f(n) = \Omega(g(n))$  means that there exists some  $M > 0$  and a sequence  $n_1 < n_2 < \dots$  such that  $f(n_j) \geq Mg(n_j)$  for all  $j$ . Additionally,  $f(n) \ll g(n)$  is equivalent to  $f(n) = O(g(n))$  and  $f(n) \gg g(n)$  is equivalent to  $g(n) \ll f(n)$ . Note that  $\ll_a$ ,  $\gg_a$ , and  $\Omega_a$  indicate that the implicit constant  $M$  is dependent on  $a$ .

Lastly, suppose that  $f(n)$  and  $g(n)$  tend to  $\infty$  with  $n$ . Then,  $f(n) \sim g(n)$  means that  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1$ .

## 2. STATEMENT OF RESULTS

We can prove Conjecture 1 on average.

**Theorem 1.** *As  $N$  goes to infinity,*

$$\frac{1}{N} \sum_{n \leq N} R_3(n) \sim \frac{1}{2} \log^2 N.$$

For an upper bound, we can show the following.

**Theorem 2.** *For any  $\epsilon > 0$ ,*

$$R_3(n) \ll_{\epsilon} n^{1/3} \log n (\log \log n)^4.$$

In the course of proving an upper bound for  $R_3(n)$ , we discovered some interesting properties of  $R_3(n)$  which we state here.

**Proposition 1.** *If  $R_3(n) = 0$  for  $n > 2$ , then  $n$  is prime and  $2n - 3$  is prime.*

*Proof.* Let

$$f_3(x, y, z) = xyz + x + y + z.$$

Observe that  $f_3(x, y, 1) = (x+1)(y+1)$ . Thus, if  $n$  is composite, there exists a triple  $(x, y, 1)$  such that  $f_3(x, y, 1) = n$ . Thus,  $n$  is prime. The second assertion is obviously true for  $n = 3$ . If  $n > 3$  and  $2n - 3$  is composite, then

$$2n - 3 = (2x + 1)(2y + 1)$$

for some positive  $x, y$ . This implies

$$n = 2xy + x + y + 2,$$

which is a solution with  $z = 2$ , which means that  $f_3(x, y, 2) = n$ . Thus,  $2n - 3$  is prime.  $\square$

This result led us to wonder how many primes  $p \leq x$  there are for which  $R_3(p) = 0$ . Define

$$U_3(N) = \#\{n \leq N : R_3(n) = 0\}.$$

Using the large sieve we can show the following.

**Theorem 3.** *There exists some constant  $c > 0$  such that*

$$U_3(N) \ll \frac{N}{e^{c\sqrt{\log N}}}$$

as  $N$  goes to infinity.

We expect that this estimate is far from optimal. We did calculations for  $N$  up to the 250 millionth prime (5336500537) and found 2014 prime numbers which are not the sum plus the product of three positive integers.

Based on numerical evidence, we do believe that  $U_3(N) \rightarrow \infty$  as  $N \rightarrow \infty$  but very slowly, maybe like a power of  $\log N$ .

**Conjecture 2.** *There exist positive constants  $A < B$  such that*

$$\log^A N \ll U_3(N) \ll \log^B N.$$

It is interesting to compare this problem with the analogous problem in four variables. Let  $R_4(n)$  denote the number of positive integral solutions to

$$n = xyzw + x + y + z + w$$

and let

$$U_4(N) = \#\{n \leq N : R_4(n) = 0\}.$$

**Conjecture 3.** *For any  $\epsilon > 0$ ,*

$$R_4(n) \ll_{\epsilon} n^{\epsilon}.$$

Again, we can prove this conjecture on average.

**Theorem 4.** *As  $N$  goes to infinity,*

$$\frac{1}{N} \sum_{n \leq x} R_4(n) \sim \frac{1}{6} \log^3 N.$$

We can also obtain an upper bound of  $R_4(n)$ .

**Theorem 5.** *For any  $\epsilon > 0$ ,*

$$R_4(n) \ll_{\epsilon} n^{\frac{1}{2} + \epsilon}.$$

By the large sieve inequality we can show the following.

**Theorem 6.** *For any  $C > 0$  we have*

$$U_4(N) \ll \frac{N}{e^{C\sqrt{\log N}}}$$

as  $N$  goes to infinity.

Unlike the situation for  $R_3(n)$ , we believe that only a finite number of  $n$  have  $R_4(n) = 0$ .

**Conjecture 4.** *If  $n > 45752$ , then*

$$R_4(n) > 0.$$

**2.1. Omega results.** If  $n \geq 5$ , by consideration of solutions where at least one of  $x, y, z$  is equal to 1, it is easy to see that

$$R_3(n) \geq 6\tau(n) - 6$$

where  $\tau(n) = \tau_2(n)$  is the number of divisors of  $n$ . Using what we know about how large  $\tau(n)$  can get, it follows that for every  $\epsilon > 0$ ,

$$(2) \quad R_3(n) = \Omega_\epsilon \left( \exp\left(\frac{\log(2 - \epsilon) \log n}{\log \log n}\right) \right).$$

One could also consider a solution where at least one of  $x, y, z$  is equal to 2, in which case we can see

$$R_3(n) \geq 6\tau(n) + 6\tau(2n - 3) + O(1).$$

It is not clear to us how big  $\tau(n) + \tau(2n - 3)$  can be. One could do similarly for 3, 4, ... and obtain apparently stronger omega results. In particular, let

$$\tau(n; q, a) = \sum_{\substack{d|n \\ d \equiv a \pmod{q}}} 1.$$

By (3) (below), we see that

$$R_3(n) \geq \sum_{1 \leq x \leq n^{\frac{1}{3}}} \max(\tau(xn - (x^2 - 1); x, 1) - 2, 0)$$

Note Proposition 1 is an immediate corollary of this.

We wonder whether

$$R_3(n) = \Omega_\epsilon \left( \exp\left(\frac{\log(3 - \epsilon) \log n}{\log \log n}\right) \right),$$

which would be the case if  $R_3(n)$  behaves like  $\tau_3(n)$ .

**2.2. Algorithm for calculating  $R_3(n)$ .** In order to find all the  $n \leq 5 \times 10^9$  with  $R_3(n) = 0$ , we need a reasonable algorithm for determining when  $R_3(n) > 0$ . Suppose that  $xyz + x + y + z = n$  and that  $x \leq y \leq z$ . Multiplying through by  $x$  and adding 1 to both sides yields

$$(3) \quad (xy + 1)(xz + 1) = nx - x^2 + 1.$$

Therefore, if we can find the proper divisors  $d$  of  $nx - x^2 + 1$  such that  $d \equiv 1 \pmod{x}$ , then from this set of divisors we can easily determine the complete set of solutions of  $f_3(x, y, z) = n$ . If there are no proper divisors  $d \equiv 1 \pmod{x}$ , then  $R_3(n) = 0$ . The probabilistic time complexity for prime factorization of a number  $m$  is  $O(\exp((\log m)^{\frac{1}{2}+\epsilon}))$ . We apply this to  $m = nx - x^2 + 1$ , for all  $x \leq n^{\frac{1}{3}}$  and so for each  $n$ , there are  $n^{\frac{1}{3}+\epsilon}$  steps to determine the value of  $R_3(n)$ .

For the primes  $p$  up to 5336500537, we checked using this method whether or not  $R_3(p) = 0$ . We computed all 2014 primes up to 5336500537 such that  $R_3(p) = 0$ :

$$2, 3, 5, 7, 11, 13, 17, 23, 31, 37, 41, 43, 53, 67, 71, 83, 97, 101, 107, 113, \\ \dots, 5178563387, 5220047297, 5284333573, 5322410117$$

See [this link](#) for the full list.

**2.3. Algorithm for calculating  $R_4(n)$ .** Let

$$f_4(x, y, z, w) = xyzw + x + y + z + w.$$

Suppose  $f_4(x, y, z, w) = n$  and  $x \leq y \leq z \leq w$ . Multiplying through by  $xy$  and rearranging yields

$$(xyz + 1)(xyw + 1) = nxy + 1 - x^2y - xy^2.$$

So, to find  $R_4(n)$ , it is a matter of finding proper divisors  $d$  of  $nxy + 1 - x^2y - xy^2$  such that  $d \equiv 1 \pmod{xy}$ . This involves  $\ll n^{\frac{1}{2}} \log n$  factorizations of numbers that are less than  $n^{\frac{3}{2}}$ . So, the total time to calculate  $R_4(n)$  is  $O(n^{\frac{1}{2}+\epsilon})$ .

Since the number of divisors of  $nxy + 1 - x^2y - xy^2$  is  $O((nxy + 1 - x^2y - xy^2)^\epsilon) = O(n^\epsilon)$ , this argument also proves Theorem 5.

Next, note that

$$f_4(x, y, z, 1) = f_3(x, y, z) + 1.$$

Therefore, if  $R_3(n) > 0$ , then  $R_4(n+1) > 0$ . Thus, to verify that  $R_4(n) > 0$  for  $45752 < n \leq 5336500538$ , it suffices to check that  $R_4(p+1) > 0$  for the 2014 values of  $p \leq 5336500537$  for which  $R_3(p) = 0$ . For each of these,  $R_4(p+1) > 0$ .

We believe that the complete list of numbers  $n$  for which  $R_4(n) = 0$  is:

$$1, 2, 3, 4, 6, 8, 12, 14, 18, 32, 38, 44, 54, \\ 68, 102, 108, 182, 192, 194, 224, 252, 374, 422, \\ 432, 908, 1092, 1202, 1278, 2468, 2768, \\ 3182, 4508, 7208, 16104, 21998, 26348, 45752$$

**2.4. Residue Classes.** The expression  $f_3(x, y, z) = xyz + x + y + z$  is also equal to

$$z(xy + 1) + x + y.$$

As a result, every positive integer  $n \equiv x + y \pmod{xy + 1}$  with  $n > xy + 1$  has  $R_3(n) > 0$ . For example, if  $x = 2$  and  $y = 2$ , we see that every  $n > 5$  such that  $n \equiv 4 \pmod{5}$  has  $R_3(n) > 0$ . Similarly, if  $n \equiv 5 \pmod{7}$ , if  $n \equiv 7 \pmod{11}$  or if  $n \equiv 7$  or  $8 \pmod{13}$ , then  $R_3(n) > 0$ . In general, there are

$$(4) \quad f_3(p) = \frac{1}{2}(\tau(p-1) - 2)^1$$

residue classes modulo any odd prime  $p$  such that if  $n$  is congruent to one of these residue classes mod  $p$ , then  $R_3(n) > 0$ .

Consequently, to give a bound on the number of  $n \leq N$  for which  $R_3(n) = 0$ , we can count how many  $n \leq N$  are not in any of these residue classes. This idea suggests using the large sieve to estimate  $U_3(n)$ , which we will do in the proof of Theorem 3.

Now, we do the same thing for  $f_4(x, y, z, w) = xyzw + x + y + z + w$ . Note that

$$xyzw + x + y + z + w = w(xyz + 1) + x + y + z.$$

Just as for  $f_3(x, y, z)$ , this shows every positive integer  $n \equiv x + y + z \pmod{xyz + 1}$  with  $n > xyz + 1$  has  $R_4(n) > 0$ . For example, if  $x = 2$ ,  $y = 2$ , and  $z = 3$ , we see that every  $n > 9$  such that  $n \equiv 7 \pmod{13}$  has  $R_4(n) > 0$ . In general, there are

$$(5) \quad f_4(p) = \frac{1}{6}(\tau_3(p-1) - 3)$$

residue classes modulo any odd prime  $p$  such that if  $n$  is congruent to one of these residue classes mod  $p$ , then  $R_4(n) > 0$ . We use this again in the proof of Theorem 6.

### 3. PROOFS

#### 3.1. Proofs of Theorems 1 and 4.

*Proof.* Observe that we can split the sum into two parts:

$$\sum_{n \leq N} R_3(n) = \sum_{xyz+x+y+z \leq N} 1 = \sum_{xyz+x+y+z \leq \frac{N}{\log N}} 1 + \sum_{\frac{N}{\log N} < xyz+x+y+z < N} 1$$

For the second part,

$$\sum_{\frac{N}{\log N} < xyz+x+y+z < N} 1 = \sum_{xy < N} \sum_{\frac{N}{\log N} - x - y \over xy + 1 < z < \frac{N - x - y}{xy + 1}} 1$$

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<sup>1</sup> $f_3(p)$  and  $f_3(x, y, z)$  denote different functions. The same is true for  $f_4(p)$  and  $f_4(x, y, z, w)$ .

If  $x + y + z \gg \frac{N}{\log N}$  and  $xyz < N$ , then  $\max(x, y, z) \gg \frac{N}{\log N}$ . Without loss of generality, let  $z$  be the maximum of  $x, y, z$ . Then,

$$N \gg xyz \gg xy \frac{N}{\log N}.$$

Therefore,

$$x + y \ll xy \ll \log N.$$

Thus,

$$\sum_{xy < N} \sum_{\substack{\frac{N}{\log N} - x - y \\ xy + 1} < z < \frac{N - x - y}{xy + 1}} 1 \ll \sum_{\substack{xyz \leq 2N \\ xy \ll \log N}} 1 \ll \sum_{xy \ll \log N} \frac{N}{xy} \ll N(\log \log N)^2$$

Therefore, the terms with  $x + y + z \gg \frac{N}{\log N}$  are negligible, and we can assume that  $x + y + z \ll \frac{N}{\log N}$ . Then, we have

$$\sum_{xyz + x + y + z < N} 1 = \sum_{xyz < N + O(\frac{N}{\log N})} 1 = \sum_{n < N + O(\frac{N}{\log N})} \tau_3(n) \sim \frac{N \log^2 N}{2}$$

(see 12.1.4 of [T]), which proves Theorem 1. □

The proof of Theorem 4 is similar.

### 3.2. Proof of Theorem 2.

*Proof.* If  $n = xyz + x + y + z$ , then  $nx - x^2 + 1 = (xy + 1)(xz + 1)$ . Therefore,

$$R_3(n) \leq 6 \sum_{x \leq n^{\frac{1}{3}}} \tau(nx - x^2 + 1).$$

We now use the following estimate, which is a consequence of Henriot ([H1] and [H2]):

$$\sum_{x < n^{\frac{1}{3}}} \tau(nx - x^2 + 1) \ll n^{\frac{1}{3}} \log n \prod_{p|n^2+4} (1 + 4/p).$$

Therefore,

$$R_3(n) \ll n^{\frac{1}{3}} \log n \prod_{p|n^2+4} (1 + 1/p)^4.$$

By the prime number theorem,

$$\prod_{p|n^2+4} (1 + 1/p) \leq \prod_{p \leq 3 \log n} (1 + 1/p).$$

And by Merten's Theorem,

$$\prod_{p \leq 3 \log n} (1 + 1/p) \ll \log \log n$$

and the result follows. □

### 3.3. Proof of Theorem 3.

*Proof.* Define

$$a_n = \begin{cases} 0 & \text{if } R_3(n) > 0 \\ 1 & \text{if } R_3(n) = 0 \end{cases}$$

and define  $Z = \sum_{n=1}^N a_n$ . Note that  $Z = U_3(N)$ . By the discussion in section 1.4 and the large sieve inequality (see Theorem 7.11 of [O]), we have that

$$(6) \quad Z \leq \frac{(N^{\frac{1}{2}} + X)^2}{Q}$$

where  $X$  is a free parameter and

$$Q = \sum_{q \leq X} \mu^2(q) \prod_{p|q} \frac{f_3(p)}{p - f_3(p)},$$

where  $\mu$  denotes the Möbius function and  $f_3(p)$  is defined as in (4). We now prove the following lemma:

**Lemma 1.** *Let*

$$f_3(p) = \frac{\tau(p-1) - 2}{2}$$

*for a prime  $p > 3$ . Then, there exists a  $c > 0$  such that as  $X \rightarrow \infty$ , we have*

$$Q := \sum_{\substack{q \leq X \\ (q,6)=1}} \mu^2(q) \prod_{p|q} \frac{f_3(p)}{p - f_3(p)} \gg \exp\left(c\sqrt{\log X}\right).$$

*Proof.* First of all

$$Q \geq \sum_{q \leq X} \mu^2(q) \prod_{p|q} \frac{f_3(p)}{p}.$$

Let  $g_3(p) = \tau(p-1)$ . Then for  $p > 3$  we have  $f_3(p) \geq \frac{g_3(p)}{6}$ . Therefore, for any  $k$ , by restricting only to squarefree integers  $q$  with  $k$  distinct prime factors, we have

$$Q \geq 6^{-k} \prod_{\substack{p_1 \cdot p_2 \cdots p_k \leq X \\ 3 < p_1 < p_2 < \cdots < p_k}} \frac{g_3(p_1) \cdots g_3(p_k)}{p_1 \cdots p_k}.$$

Now the Titchmarsh divisor problem (as found in Theorem 3.9 of [HR]) asserts that

$$\sum_{3 < p \leq X} \frac{\tau(p-1)}{p} \sim C \log X$$



which implies that for any  $0 < a < b$ ,

$$\sum_{X^a < p \leq X^b} \frac{\tau(p-1)}{p} \sim C(b-a) \log X.$$

Therefore,

$$Q \geq 6^{-k} \sum_{3 < p_1 < X^{\frac{1}{k^2}}} \sum_{X^{\frac{1}{k^2}} < p_2 < X^{\frac{2}{k^2}}} \cdots \sum_{X^{\frac{k-1}{k^2}} < p_k < X^{\frac{k}{k^2}}} \frac{g_3(p_1) \cdots g_3(p_k)}{p_1 \cdots p_k} \gg \left(\frac{C}{6k^2} \log X\right)^k$$

for  $X \geq 10$  and any  $k < \sqrt{\frac{\log X}{\log 3}}$ . Now we choose

$$k = b\sqrt{\log X}$$

with a sufficiently small  $b$  and have

$$Q \gg \exp\left(\sqrt{\log X} \left(b \log \frac{C}{6b^2}\right)\right) \gg \exp(c\sqrt{\log X})$$

for some small  $c > 0$  as claimed. □

Using the result of Lemma 1 in (6) and choosing  $X = N^{\frac{1}{2}}$ , we obtain the bound

$$U_3(N) \ll \frac{N}{e^{c\sqrt{\log N}}}.$$

□

**3.4. Proof of Theorem 6.** The proof is similar to the proof of Theorem 3 except that we use

$$\sum_{3 < p \leq X} \frac{\tau_3(p-1)}{p} \sim C' \log^2 X$$

and  $f_4(p)$  as defined in (5). We leave the details to the reader.

#### 4. CONCLUSION AND OPEN QUESTIONS

Our proofs for bounds on  $U_3(N)$  and  $U_4(N)$  are sieve problems with two different numbers of residue classes.  $U_3(N)$  has  $c \log p$  residue classes per prime  $p$  on average that get sieved out and  $U_4(N)$  has  $c \log^2 p$  residue classes per prime  $p$  on average that get sieved out. It is interesting to note that  $U_3(N)$  appears to go to infinity and  $U_4(N)$  appears to be bounded.

Another problem which bears similarity is expressing  $n$  as the sum of three positive integer cubes. Similar to the proof of Theorem 2, one can obtain  $r_3(n) \ll n^{\frac{1}{3}} \log n (\log \log n)^a$  for some  $a$ . By Hooley's paucity results, one can obtain  $r_3(n) \ll n^{\frac{1}{3}}$ . Hardy and Littlewood's [HL] Hypothesis K was that  $r_3(n) \ll n^\epsilon$ . However, in 1936, Mahler found a parametric family of solutions for  $n$ 's which are perfect twelfth powers and proved that  $r_3(n) = \Omega(n^{\frac{1}{12}})$ .

Here are some related problems we think are worth exploring further:

- (1) Prove Conjecture 1, that  $R_3(n)$  grows slower than  $n^\epsilon$  for any  $\epsilon > 0$ .
- (2) Prove Conjecture 4.
- (3) Prove that there are infinitely many positive integers  $n$  for which  $R_3(n) = 0$ .
- (4) Does  $R_4(n)$  go to  $\infty$  with  $n$ ?
- (5) Give any improvement on the omega result in (2).
- (6) Give omega results for  $\tau(n) + \tau(n + 1)$  and  $\tau(n^2 + 1)$ .

We hope that this work stimulates the reader to pursue some of these questions further.

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