CHARACTER SUMS AND THE RIEMANN HYPOTHESIS

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In this paper we show that a simple inequality for real character sums implies the Riemann Hypothesis. Specifically,

Theorem 1. *Let $q > 0$ be squarefree with $q \equiv 3 \mod 4$ and let $h(q)$ be the class number of the imaginary quadratic field $K = Q(\sqrt{-q})$. Let χ_q be the Jacobi symbol modulo q so that χ_q is the quadratic character associated with K. Suppose that*

$$S_q(N) := \sum_{n=1}^{N} \chi_q(n)\big(1 - \tfrac{n}{N}\big) \leqslant h(q) = S_q((q-1)/2)$$

for all q as described above and all $N < \frac{q}{4}$. Then all complex zeros of the Riemann zeta-function have real part equal to 1/2.

We have tested this inequality and found it to hold for all relevant q up to 5000. We believe that it may be true in general. Note that equality holds when $N = (q-1)/2$.

The basic idea of the proof is straightforward. We consider the Fourier series

$$f(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\lambda(n) \sin 2\pi n x}{n^2}$$

where λ is the Liouville function defined by

$$\lambda(n) = (-1)^{\Omega(n)}$$

where $\Omega(n)$ is the total number of prime factors of n counted with multiplicity. Note that the series defining f is absolutely convergent. Thus, f is an odd, continuous, real function of a real variable which is periodic with period 1. We find an explicit formula for f in terms of simple functions and a sum over zeros of $\zeta(s)$. Let $\ell(n)$ be defined through its generating function

$$\sum_{n=1}^{\infty} \ell(n) n^{-s} = \frac{\zeta(2s-1)}{\zeta(s)}$$

for $\Re s > 1$.

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Theorem 2. *Suppose that all of the zeros of the Riemann zeta-function are simple and that*

$$\zeta'(\rho) \gg |\rho|^{\delta-1}$$

for some $\delta > 0$. Then, for $x \geqslant 0$,

$$f(x) = -\frac{4\pi x^{3/2}}{3\zeta(1/2)} - \frac{3\pi}{2}x^{3/2}\sum_{n\leqslant 4x}\frac{\ell(n)}{\sqrt{n}}\left(1-\frac{n}{4x}\right)^{3/2} + \sum_{\rho}\frac{X(2-\rho)\zeta(2\rho)x^{2-\rho}}{(2-\rho)\zeta'(\rho)}$$

where $X(s)$ is the factor from the functional equation for $\zeta(s)$ which can be defined by

$$X(s)^{-1} = X(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\tfrac{\pi s}{2}.$$

The sum over zeros is absolutely convergent for all real x.

From the proof of Theorem 2 we deduce

Theorem 3. *Suppose that there exists an $\eta > 0$ such that $f(x) \geqslant 0$ for $0 \leqslant x \leqslant \eta$. Then the Riemann Hypothesis is true. Moreover, if the Riemann Hypothesis is true and*

$$\sum_{\rho}\left|\frac{X(2-\rho)\zeta(2\rho)}{(2-\rho)\zeta'(\rho)}\right| \le -\frac{4\pi}{3\zeta(1/2)}$$

then

$$f(x) \geqslant 0$$

for $0 \leqslant x < 1/4$.

Note that

$$-\frac{4\pi}{3\zeta(1/2)} = 2.86834\ldots$$

and

$$\sum_{|\gamma|\le 1000}\left|\frac{X(2-\rho)\zeta(2\rho)}{(2-\rho)\zeta'(\rho)}\right| = 0.264954\ldots$$

so that the inequality of the second part of Theorem 3 is quite likely to be valid. The reason that we state the inequality for $x < 1/4$ is that for this range of x the sum involving $\ell(n)$ is empty. In fact, $f(1/2) = 0$ and it is conceivable that for some x near $1/2$ it could be the case that $f(x) < 0$.

The idea for the proof of Theorem 1 is to approximate $\lambda(n)$ by a Dirichlet character $\chi(n)$. If there is an x with $f(x) < 0$, then it is not difficult to see that there will be an odd real Dirichlet character χ modulo q such that

$$f_\chi(x) := \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\chi(n)\sin 2\pi n x}{n^2} < 0.$$

To see this, we observe that by the Chinese Remainder theorem and quadratic reciprocity we can find a modulus q such that

$$\left(\frac{p}{q}\right) = -1$$

for any initial segment of primes $p \leqslant m(x)$; then $\lambda(n) = \left(\frac{n}{q}\right) = \chi_q(n)$ for $n \leqslant m(x)$. If $m(x)$ is chosen sufficiently large, then $f(x) < 0$ implies that $f_\chi(x) < 0$.

An explicit formula for f_χ is given by

Theorem 4. *Let $x \geqslant 0$. Let χ be an odd real, primitive character modulo q. Then*

$$f_\chi(x) = 2xL(1,\chi) - \frac{2\pi x}{\sqrt{q}} \sum_{n \leqslant xq} \chi(n)\left(1 - \tfrac{n}{xq}\right).$$

It follows from Theorem 4 that the inequality $f_\chi(x) \geqslant 0$ is equivalent to (with $N = xq$)

$$\sum_{n \leqslant N} \chi(n)\left(1 - \tfrac{n}{N}\right) \leqslant \frac{1}{\pi} L(1,\chi)\sqrt{q} = h(q)$$

the equality at the end following from Dirichlet's class number formula. Then Theorem 1 follows.

Now we proceed to the proofs of Theorems 3 and 4 and then deduce Theorem 2.

Lemma 1. *For $y > 0$ we have*

$$\frac{1}{2\pi i} \int_{(c)} \frac{X(1-s) y^{1-s}}{1-s}\, ds = \frac{\sin 2\pi y}{\pi}$$

for any c satisfying $0 < c < 1$ where (c) denotes the path from $c - i\infty$ to $c + i\infty$

This lemma is quite believable since the integrand has simple poles at $s = 0, -2, -4, \ldots$ $\blacksquare$ with the residue at $s = -2n$ equal to

$$\frac{1}{\pi} \frac{(-1)^n (2\pi y)^{2n+1}}{(2n+1)!}.$$

For a proof of this Lemma see [T]; the above is the integral of the formula (7.9.5).

Lemma 2. *If $c > 0$ and $\Re a > 0$, then*

$$\frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s)\Gamma(a)}{\Gamma(s+a)} x^{-s}\, ds = \begin{cases} (1-x)^{a-1} & \text{if } 0 < x < 1 \\ 0 & \text{if } x \geqslant 1 \end{cases}$$

This formula is (7.9.6) of [T].

Lemma 3. *If $c > 0$, then*

$$\frac{1}{2\pi i} \int \frac{x^s}{s(s+1)}\, ds = \begin{cases} 1 - \frac{1}{x} & \text{if } x > 1 \\ 0 & \text{if } 0 < x \leqslant 1 \end{cases}$$

This lemma is well-known and is easy to verify.

Proof of Theorem 2. Since

$$\sum_{n=1}^{\infty} \lambda(n) n^{-s} = \frac{\zeta(2s)}{\zeta(s)}$$

it follows from the Lemma that

$$f(x) = \frac{1}{2\pi i} \int_{(c)} \frac{X(1-s)\zeta(2s+2)x^{1-s}}{(1-s)\zeta(s+1)}\, ds$$

where $0 < c < 1/2$. The integrand has poles only at $s = -\frac{1}{2}$ and at $s = \rho - 1$ where ρ is a complex zero of $\zeta(s)$ and nowhere else in the s-plane. The residue at $s = -\frac{1}{2}$ is

$$\frac{X(\frac{3}{2})}{\frac{3}{2}\zeta(\frac{1}{2})} x^{3/2} = -\frac{4\pi}{3\zeta(\frac{1}{2})} x^{3/2}.$$

Assuming that the zeros are simple, the residue at $s = \rho - 1$ is

$$\frac{X(2-\rho)\zeta(2\rho)x^{2-\rho}}{(2-\rho)\zeta'(\rho)}.$$

We move the path of integration to (c) where $-2 < c < -1$. Thus,

$$f(x) = -\frac{4\pi x^{3/2}}{3\zeta(1/2)} + \sum_{\rho} \frac{X(2-\rho)\zeta(2\rho)x^{2-\rho}}{(2-\rho)\zeta'(\rho)} + \frac{1}{2\pi i} \int_{(c)} \frac{X(1-s)\zeta(2s+2)x^{1-s}}{(1-s)\zeta(s+1)}\, ds$$

We make the change of variable $s \to -s$ in the integral. Using the functional equation for the ζ-function and functional relations for the Γ-function, we see that the new integrand is

$$\frac{X(1+s)\zeta(2-2s)x^{1+s}}{(1+s)\zeta(1-s)} = \pi^{3/2} 2^{2s} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+2)} \frac{\zeta(2s-1)}{\zeta(s)} x^{1+s}.$$

By Lemma 2,

$$\frac{1}{2\pi i} \int_{(c)} \pi^{3/2} 2^{2s} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+2)} \frac{\zeta(2s-1)}{\zeta(s)} x^{1+s} = \frac{3\pi}{2} x^{3/2} \sum_{n \leqslant 4x} \frac{\ell(n)}{\sqrt{n}} \left(1 - \frac{n}{4x}\right)^{3/2}.$$

The theorem follows.

Proof of Theorem 3. Let $g(T) = f(1/T)$. By Mellin inversion of the formula at the beginning of the last proof

$$G(s) = \frac{X(1-s)\zeta(2s+2)T^{s-1}}{(1-s)\zeta(s+1)} = \int_0^{\infty} g(T) T^{-s}\, dT$$

for $\Re s > 0$. Now $|g(T)| \leqslant \frac{\pi^2}{6}$ for all T. Therefore, the part of the integral from 0 to $1/\eta$ is an entire function of s. By hypothesis, $g(T) \geqslant 0$ for all $T \geqslant 1/\eta$. Therefore, by Landau's Theorem (see [HR], Theorem 10),

$$G(s) = \int_0^{\infty} g(T) T^{-s}\, dT$$

must have a pole on the real axis at the abscissa of absolute convergence. But the only real pole of $G(s)$ is at $s = -1/2$. Therefore, the integral is absolutely convergent in $\Re s > -1/2$, and consequently has no poles in this region. Therefore, $G(s)$ is analytic in $\Re s > -1/2$. Clearly, $G(s)$ has a pole at $s = \rho - 1$ for each zero ρ of $\zeta(s)$. Therefore, there are no ρ with $\Re\rho > 1/2$, i.e. the Riemann Hypothesis.

Proof of Theorem 4. We assume that $\chi = \chi_q$ for a squarefree $q \equiv 3 \mod 4$. By Lemma 1,

$$f_\chi(x) = \frac{1}{2\pi i}\int_{(c)} L(s+1,\chi)X(1-s)x^{1-s}\,\frac{ds}{1-s}$$

where $0 < c < 1$. Since χ is odd, we find that the integrand has a pole at $s = 0$ and nowhere else in the complex plane. We move the path of integration to (c) where $c < -1$ to see that

$$f_\chi(x) = 2xL(1,\chi) + \frac{1}{2\pi i}\int_{(c)} L(s+1,\chi)X(1-s)x^{1-s}\,\frac{ds}{1-s}.$$

Now let $s \to -s$ in the integral and use the functional equation

$$L(1-s,\chi) = 2q^{s-\frac{1}{2}}(2\pi)^{-s}\Gamma(s)\sin\tfrac{\pi s}{2}L(s,\chi).$$

After simplification, the integral above is

$$\frac{-2\pi}{2\pi i}\int_{(c)} q^{s-\frac{1}{2}}x^{1+s}L(s,\chi)\,\frac{ds}{s(s+1)}.$$

By Lemma 3, this integral is

$$\frac{-2\pi x}{\sqrt{q}}\sum_{n\leqslant xq}\chi(n)\left(1-\frac{n}{xq}\right).$$

The proof of the Theorem is complete.

Remark. For $x \leq 1/q$ in Theorem 4, we see that

$$F_\chi(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\chi(n)\sin 2\pi nx}{n^2} = 2xL(1,\chi)$$

is a non-negative linear function.

Also, since

$$h(q) \gg_\epsilon q^{1/2-\epsilon}$$

we see that our desired inequality is (trivially) always true for $N \ll_\epsilon q^{1/2-\epsilon}$. Also, the Polya-Vinogradov inequaliy tells us that

$$\max_N \Big|\sum_{n=1}^{N}\chi_q(n)\Big| \ll q^{1/2}\log q$$

and the work of Montgomery and Vaughan [MV] shows that the Riemann Hypothesis for $L(s,\chi)$ implies that

$$\max_N \Big|\sum_{n=1}^{N} \chi_q(n)\Big| \ll q^{1/2}\log\log q.$$

Moreover, it is known that the right hand side here can not be replaced by any function that goes to infinity slower. It is also known, assuming the Riemann Hypothesis for $L(s,\chi)$, that

$$L(1,\chi) \ll \log\log q.$$

Our desired inequality can be expressed in terms of $L(1,\chi)$ as

$$\max_{N\leqslant \frac{q}{4}} \chi(n)\big(1-\frac{n}{N}\big) \leqslant \frac{\sqrt{q}}{\pi} L(1,\chi).$$

It appears that both sides of the inequality can be as big as $\sqrt{q}\log\log q$.

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