CHARACTER SUMS AND THE Riemann Hypothesis

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In this paper we show that a simple inequality for real character sums implies the Riemann Hypothesis. Specifically,

**Theorem 1.** Let \( q > 0 \) be squarefree with \( q \equiv 3 \mod 4 \) and let \( h(q) \) be the class number of the imaginary quadratic field \( K = \mathbb{Q}(\sqrt{-q}) \). Let \( \chi_q \) be the Jacobi symbol modulo \( q \) so that \( \chi_q \) is the quadratic character associated with \( K \). Suppose that

\[
S_q(N) := \sum_{n=1}^{N} \chi_q(n)(1 - \frac{n}{N}) \leq h(q) = S_q((q-1)/2)
\]

for all \( q \) as described above and all \( N < \frac{q}{4} \). Then all complex zeros of the Riemann zeta-function have real part equal to 1/2.

We have tested this inequality and found it to hold for all relevant \( q \) up to 5000. We believe that it may be true in general. Note that equality holds when \( N = (q-1)/2 \).

The basic idea of the proof is straightforward. We consider the Fourier series

\[
f(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\lambda(n) \sin 2\pi nx}{n^2}
\]

where \( \lambda \) is the Liouville function defined by

\[
\lambda(n) = (-1)^{\Omega(n)}
\]

where \( \Omega(n) \) is the total number of prime factors of \( n \) counted with multiplicity. Note that the series defining \( f \) is absolutely convergent. Thus, \( f \) is an odd, continuous, real function of a real variable which is periodic with period 1. We find an explicit formula for \( f \) in terms of simple functions and a sum over zeros of \( \zeta(s) \). Let \( \ell(n) \) be defined through its generating function

\[
\sum_{n=1}^{\infty} \ell(n)n^{-s} = \frac{\zeta(2s-1)}{\zeta(s)}
\]

for \( \Re s > 1 \).

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Theorem 2. Suppose that all of the zeros of the Riemann zeta-function are simple and that
\[ \zeta'(\rho) \gg |\rho|^{\delta-1} \]
for some \( \delta > 0 \). Then, for \( x \geq 0 \),
\[
f(x) = -\frac{4\pi x^{3/2}}{3\zeta(1/2)} - \frac{3\pi}{2} x^{3/2} \sum_{n \leq 4x} \frac{\ell(n)}{\sqrt{n}} \left(1 - \frac{n}{4x}\right)^{3/2} + \sum_{\rho} \frac{X(2 - \rho)\zeta(2\rho)x^{2-\rho}}{(2 - \rho)\zeta'(\rho)}
\]
where \( X(s) \) is the factor from the functional equation for \( \zeta(s) \) which can be defined by
\[ X(s)^{-1} = X(1 - s) = 2(2\pi)^{-s}\Gamma(s)\cos\frac{\pi s}{2}. \]
The sum over zeros is absolutely convergent for all real \( x \).

From the proof of Theorem 2 we deduce

Theorem 3. Suppose that there exists an \( \eta > 0 \) such that \( f(x) \geq 0 \) for \( 0 \leq x \leq \eta \). Then the Riemann Hypothesis is true. Moreover, if the Riemann Hypothesis is true and
\[
\sum_{\rho} \left| \frac{X(2 - \rho)\zeta(2\rho)}{(2 - \rho)\zeta'(\rho)} \right| \leq -\frac{4\pi}{3\zeta(1/2)}
\]
then
\[ f(x) \geq 0 \]
for \( 0 \leq x < 1/4 \).

Note that
\[ -\frac{4\pi}{3\zeta(1/2)} = 2.86834 \ldots \]
and
\[
\sum_{|\gamma| \leq 1000} \left| \frac{X(2 - \rho)\zeta(2\rho)}{(2 - \rho)\zeta'(\rho)} \right| = 0.264954 \ldots
\]
so that the inequality of the second part of Theorem 3 is quite likely to be valid.

The reason that we state the inequality for \( x < 1/4 \) is that for this range of \( x \) the sum involving \( \ell(n) \) is empty. In fact, \( f(1/2) = 0 \) and it is conceivable that for some \( x \) near 1/2 it could be the case that \( f(x) < 0 \).

The idea for the proof of Theorem 1 is to approximate \( \lambda(n) \) by a Dirichlet character \( \chi(n) \). If there is an \( x \) with \( f(x) < 0 \), then it is not difficult to see that there will be an odd real Dirichlet character \( \chi \) modulo \( q \) such that
\[
f_\chi(x) := \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\chi(n)\sin 2\pi nx}{n^2} < 0.
\]
To see this, we observe that by the Chinese Remainder theorem and quadratic reciprocity we can find a modulus \( q \) such that
\[ \left( \frac{p}{q} \right) = -1 \]
for any initial segment of primes \( p \leq m(x) \); then \( \lambda(n) = \left( \frac{n}{q} \right) = \chi_q(n) \) for \( n \leq m(x) \).

If \( m(x) \) is chosen sufficiently large, then \( f(x) < 0 \) implies that \( f_\chi(x) < 0 \).

An explicit formula for \( f_\chi \) is given by

**Theorem 4.** Let \( x \geq 0 \). Let \( \chi \) be an odd real, primitive character modulo \( q \). Then

\[
f_\chi(x) = 2xL(1, \chi) - \frac{2\pi x}{\sqrt{q}} \sum_{n \leq xq} \chi(n) \left( 1 - \frac{n}{xq} \right).
\]

It follows from Theorem 4 that the inequality \( f_\chi(x) \geq 0 \) is equivalent to (with \( N = xq \))

\[
\sum_{n \leq N} \chi(n) \left( 1 - \frac{n}{N} \right) \leq \frac{1}{\pi} L(1, \chi) \sqrt{q} = h(q)
\]

the equality at the end following from Dirichlet’s class number formula. Then Theorem 1 follows.

Now we proceed to the proofs of Theorems 3 and 4 and then deduce Theorem 2.

**Lemma 1.** For \( y > 0 \) we have

\[
\frac{1}{2\pi i} \int_{(c)} \frac{X(1-s)y^{1-s}}{1-s} \, ds = \frac{\sin 2\pi y}{\pi}
\]

for any \( c \) satisfying \( 0 < c < 1 \) where \( (c) \) denotes the path from \( c - i\infty \) to \( c + i\infty \)

This lemma is quite believable since the integrand has simple poles at \( s = 0, -2, -4, \ldots \) with the residue at \( s = -2n \) equal to

\[
\frac{1}{\pi} \frac{(-1)^n (2\pi y)^{2n+1}}{(2n+1)!}.
\]

For a proof of this Lemma see [T]; the above is the integral of the formula (7.9.5).

**Lemma 2.** If \( c > 0 \) and \( \Re a > 0 \), then

\[
\frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s)\Gamma(a)}{\Gamma(s+a)} x^{-s} \, ds = \begin{cases} (1-x)^{a-1} & \text{if } 0 < x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}
\]

This formula is (7.9.6) of [T].

**Lemma 3.** If \( c > 0 \), then

\[
\frac{1}{2\pi i} \int \frac{x^s}{s(s+1)} \, ds = \begin{cases} 1 - \frac{1}{x} & \text{if } x > 1 \\ 0 & \text{if } 0 < x \leq 1 \end{cases}
\]

This lemma is well-known and is easy to verify.
Proof of Theorem 2. Since
\[ \sum_{n=1}^{\infty} \lambda(n)n^{-s} = \frac{\zeta(2s)}{\zeta(s)} \]
it follows from the Lemma that
\[ f(x) = \frac{1}{2\pi i} \int_{(c)} \frac{X(1-s)\zeta(2s+2)x^{1-s}}{(1-s)\zeta(s+1)} \, ds \]
where \( 0 < c < 1/2 \). The integrand has poles only at \( s = -\frac{1}{2} \) and at \( s = \rho - 1 \) where \( \rho \) is a complex zero of \( \zeta(s) \) and nowhere else in the \( s \)-plane. The residue at \( s = -\frac{1}{2} \) is
\[ X\left(\frac{3}{2}\right)x^{3/2} = -\frac{4\pi}{3\zeta(3/2)}x^{3/2}. \]
Assuming that the zeros are simple, the residue at \( s = \rho - 1 \) is
\[ \frac{X(2-\rho)\zeta(2\rho)x^{2-\rho}}{(2-\rho)\zeta'(\rho)}. \]
We move the path of integration to \( (c) \) where \( -2 < c < -1 \). Thus,
\[ f(x) = -\frac{4\pi x^{3/2}}{3\zeta(1/2)} + \sum_{\rho} \frac{X(2-\rho)\zeta(2\rho)x^{2-\rho}}{(2-\rho)\zeta'(\rho)} + \frac{1}{2\pi i} \int_{(c)} \frac{X(1-s)\zeta(2s+2)x^{1-s}}{(1-s)\zeta(s+1)} \, ds \]
We make the change of variable \( s \rightarrow -s \) in the integral. Using the functional equation for the \( \zeta \)-function and functional relations for the \( \Gamma \)-function, we see that the new integrand is
\[ \frac{X(1+s)\zeta(2-2s)x^{1+s}}{(1+s)\zeta(1-s)} = \pi^{3/2}2^{2s} \frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s+2)\zeta(s)}x^{1+s}. \]
By Lemma 2,
\[ \frac{1}{2\pi i} \int_{(c)} \pi^{3/2}2^{2s} \frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s+2)\zeta(s)}x^{1+s} = \frac{3\pi}{2}x^{3/2} \sum_{n \leq 4x} \frac{\ell(n)}{\sqrt{n}} \left(1 - \frac{n}{4x}\right)^{3/2}. \]
The theorem follows.

Proof of Theorem 3. Let \( g(T) = f(1/T) \). By Mellin inversion of the formula at the beginning of the last proof
\[ G(s) = \frac{X(1-s)\zeta(2s+2)T^{s-1}}{(1-s)\zeta(s+1)} = \int_{0}^{\infty} g(T)T^{-s} \, dT \]
for \( \Re s > 0 \). Now \( |g(T)| \leq \frac{\pi^2}{6} \) for all \( T \). Therefore, the part of the integral from 0 to \( 1/\eta \) is an entire function of \( s \). By hypothesis, \( g(T) \geq 0 \) for all \( T \geq 1/\eta \). Therefore, by Landau’s Theorem (see [HR], Theorem 10),
\[ G(s) = \int_{0}^{\infty} g(T)T^{-s} \, dT \]
must have a pole on the real axis at the abscissa of absolute convergence. But the only real pole of \( G(s) \) is at \( s = -1/2 \). Therefore, the integral is absolutely convergent in \( \Re s > -1/2 \), and consequently has no poles in this region. Therefore, \( G(s) \) is analytic in \( \Re s > -1/2 \). Clearly, \( G(s) \) has a pole at \( s = \rho - 1 \) for each zero \( \rho \) of \( \zeta(s) \). Therefore, there are no \( \rho \) with \( \Re \rho > 1/2 \), i.e. the Riemann Hypothesis.

**Proof of Theorem 4.** We assume that \( \chi = \chi_q \) for a squarefree \( q \equiv 3 \mod 4 \). By Lemma 1,

\[
f_\chi(x) = \frac{1}{2\pi i} \int_{(c)} L(s + 1, \chi) X(1 - s) x^{1 - s} \frac{ds}{1 - s},
\]

where \( 0 < c < 1 \). Since \( \chi \) is odd, we find that the integrand has a pole at \( s = 0 \) and nowhere else in the complex plane. We move the path of integration to \( (c) \) where \( c < -1 \) to see that

\[
f_\chi(x) = 2x L(1, \chi) + \frac{1}{2\pi i} \int_{(c)} L(s + 1, \chi) X(1 - s) x^{1 - s} \frac{ds}{1 - s}.
\]

Now let \( s \to -s \) in the integral and use the functional equation

\[
L(1 - s, \chi) = 2q^{s - 1/2} (2\pi)^{-s} \Gamma(s) \sin \frac{\pi s}{2} L(s, \chi).
\]

After simplification, the integral above is

\[
- \frac{2\pi}{2\pi i} \int_{(c)} q^{s - 1/2} x^{1 + s} L(s, \chi) \frac{ds}{s(s + 1)}.
\]

By Lemma 3, this integral is

\[
- \frac{2\pi x}{\sqrt{q}} \sum_{n \leq xq} \chi(n) \left( 1 - \frac{n}{xq} \right).
\]

The proof of the Theorem is complete.

**Remark.** For \( x \leq 1/q \) in Theorem 4, we see that

\[
F_\chi(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\chi(n) \sin 2\pi nx}{n^2} = 2x L(1, \chi)
\]

is a non-negative linear function.

Also, since

\[
h(q) \gg \epsilon q^{1/2 - \epsilon}
\]

we see that our desired inequality is (trivially) always true for \( N \ll \epsilon q^{1/2 - \epsilon} \). Also, the Polya-Vinogradov inequality tells us that

\[
\max_N \left| \sum_{n=1}^{N} \chi_q(n) \right| \ll q^{1/2} \log q
\]
and the work of Montgomery and Vaughan [MV] shows that the Riemann Hypothesis for \( L(s, \chi) \) implies that

\[
\max_N \left| \sum_{n=1}^N \chi(q(n)) \right| \ll q^{1/2} \log \log q.
\]

Moreover, it is known that the right hand side here can not be replaced by any function that goes to infinity slower. It is also known, assuming the Riemann Hypothesis for \( L(s, \chi) \), that

\[
L(1, \chi) \ll \log \log q.
\]

Our desired inequality can be expressed in terms of \( L(1, \chi) \) as

\[
\max_{N \leq q/4} \chi(n) \left( 1 - \frac{n}{N} \right) \leq \frac{\sqrt{q}}{\pi} L(1, \chi).
\]

It appears that both sides of the inequality can be as big as \( \sqrt{q} \log \log q \).

**References**


