Notes on eigenvalue distributions for the classical compact groups

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Contents

1 Some notation

Before we get started we record some notation that will be used in these notes. This section is merely to serve as a convenient reference; the notions are defined at the appropriate places in the notes.

- The sine ratios are:

$$S(x) = \frac{\sin \pi x}{\pi x}$$

$$S_N(x) = \frac{\sin Nx/2}{\sin x/2}$$

- $G(N)$ stands for one of the groups $U(N)$, $USp(2N)$, $SO(2N)$, $SO(2N+1)$ and G by itself stands for one of the symmetry types U (unitary), Sp (symplectic), O (orthogonal) even, O (orthogonal) odd

- The kernel functions are

$$K_{U(N)}(x,y) = S_N(y-x)$$

$$K_{SO(2N)}(x,y) = \frac{S_{2N-1}(y-x) + S_{2N-1}(y+x)}{2}$$

$$K_{USp(2N)}(x,y) = \frac{S_{2N+1}(y-x) - S_{2N+1}(y+x)}{2}$$

$$K_{SO(2N+1)}(x,y) = \frac{S_{2N}(y-x) - S_{2N}(y+x)}{2}.$$

- The scaled limit of these kernel functions are

$$K_U(x,y) = S(y-x)$$

$$K_{Sp}(x,y) = S(y-x) - S(y+x)$$

$$K_{O,even}(x,y) = S(y-x) + S(y+x)$$

$$K_{O,odd}(x,y) = S(y-x) - S(y+x)$$

- For an interval J, the integral operator $K_{J,G(N)}$ is defined by

$$(K_{J,G(N)}f)(x) \quad = \quad \int_J K_{G(N)}(x,y)f(y)\;dy$$

for functions f integrable on J, and similarly the operator $K_{J,G}$ is defined by

$$(K_{J,G}f)(x) \quad = \quad \int_J K_G(x,y)f(y)\;dy$$

These operators have eigenvalues denoted by $\lambda_{j,G(N)}(J)$ $(j = 1,2,\dots,N)$ and $\lambda_{j,G}(J)$, $(j = 1,2,3,\dots)$ respectively.

- The Chebyshev polynomials are $T_n(x), U_n(x)$, and $V_n(x)$ where

$$T_n(\cos\theta) = \cos n\theta$$

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

$$V_n(\cos\theta) = U_{2n}\big(\cos\frac{\theta}{2}\big) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}.$$

- We let $\mu_{G(N),j}(s)$ be the density function for the jth nearest neighbor spacing for eigenangles of $G(N)$ and $\mu_{G,j}(s)$ is the large–N–scaled–limit of this density function. Similarly, $\nu_{G(N),j}(s)$ is the density of the jth lowest eigenangle for $G(N)$ and $\nu_{G,j}(s)$ is its scaled limit.

- We let $E_{G(N)}(J,n)$ be the measure of the set of matrices $X \in G(N)$ which have precisely n eigenangles in the set J.

2 Introduction

In 1972 the fortuitous introduction of Montgomery and Dyson served also as an introduction of the worlds of analytic number theory and random matrix theory. The symbiosis between these two subjects developed slowly for the next 25 years with the principal developments being the numerical work of Odlyzko and the calculations of the third and higher correlations of the Riemann zeta-function (and other L-functions) by Hejhal, and Rudnick-Sarnak.

Around 1998, there were two very important developments that have stimulated a great deal of subsequent work. One was the theory of symmetry types associated to families of L-functions by Katz and Sarnak. The other was the relationship between moments of characteristic polynomials and moments of the Riemann zeta-function and of families of L-functions found by Keating and Snaith.

While we still do not understand why there is such a strong connnection between random matrix theory and families of L-functions, we do realize that random matrix theory provides models for a wide range of statistical behavior of these families. Consequently, we can now confidently predict the answer to any number of difficult questions about L-functions which 10 years ago seemed hopelessly impossible.

The purpose of these notes is to provide an introduction to the random matrix aspects of the book [KaSa] by Katz and Sarnak on symmetry types associated with families of L-functions. In particular, we will develop here some of the basic tools needed to understand the beginnings of computing statistics of eigenvalues of unitary, orthogonal, and symplectic groups of matrices. The four statistics we are interested in computing are n-correlation, n–level density, jth nearest neighbor, and jth lowest eigenvalue.

The main goals of these notes are (a) to show how to rewrite the basic Weyl integration formula for each of our groups $G(N)$ as a determinant of a "kernel" function $K_{G(N)}$ (derived in sections 2 – 5, equations (9), (18), (19), and (20)); (b) to use Gaudin's lemma to compute level densities and correlations (derived in sections 6 – 8, equations (33), (34), and (35)); (c) to use the combinatorial identity (39) to deduce the mth nearest neighbor statistic from the correlations (derived in section 9.1, equation (40)); and (d) to use Gram's identity to write the neighbor and lowest eigenvalue statistics in terms of derivatives of infinite products of eigenvalues of simple operators (derived in sections 9.2 – 9.5, equations (52) and (55)).

3 Definitions and Haar measures

3.1 Unitary

If X is an $N \times N$ matrix with complex entries $X = (x_{jk})$, we let X^* be its conjugate transpose, i.e. $X^* = (x^*_{jk})$ where $x^*_{jk} = \overline{x_{kj}}$. X is said to be unitary if $XX^* = I$. We let $U(N)$ denote the group of all $N \times N$ unitary matrices. This is a compact Lie group and has a Haar measure which allows us to do analysis.

All of the eigenvalues of $X \in U(N)$ have absolute value 1; we write them as

$$e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_N}$$

with

$$0 \le \theta_j < 2\pi. \tag{1}$$

The eigenvalues of X^* are $e^{-i\theta_1}, \ldots, e^{-i\theta_N}$. Clearly, the determinant, $\det X = \prod_{n=1}^N e^{i\theta_n}$ of a unitary matrix is a complex number with absolute value equal to 1.

For any sequence of N points on the unit circle there are matrices in $U(N)$ with these points as eigenvalues. The collection of all matrices with the same set of eigenvalues constitutes a conjugacy class in $U(N)$. Thus, the set of conjugacy classes can be identified with the collection of sequences of N points on the unit circle.

We are interested in computing various statistics about these eigenvalues. Consequently, we identify all matrices in $U(N)$ that have the same set of eigenvalues. Weyl's integration formula gives a simple way to perform averages over $U(N)$ for functions f that are constant on conjugacy classes. Such functions are called 'class functions'. Note that f being constant on conjugacy classes entails that $f(\theta_1, \ldots, \theta_N)$ is necessarily symmetric in its N variables. Weyl's formula asserts that for such an f,

$$\int_{U(N)} f(X)\, dX = \int_{[0,2\pi]^N} f(\theta_1, \ldots, \theta_N) \prod_{1 \le j < k \le N} \left| e^{i\theta_k} - e^{i\theta_j} \right|^2 \frac{d\theta_1 \ldots d\theta_N}{N!(2\pi)^N}.$$

Notice that we have used X to represent a variable element of $U(N)$ and dX to denote the Haar measure. If we want to emphasize the group $U(N)$ we will designate the Haar measure by $dX_{U(N)}$.

3.2 Orthogonal and Symplectic

A unitary matrix X is said to be *orthogonal* if $XX^t = I$, where X^t denotes the transpose of X. Orthogonality for a unitary matrix implies that $X^t = X^*$ or $\overline{X} = X$. In other words any real unitary matrix is orthogonal. We let $SO(N)$ denote the subgroup of $U(N)$ consisting of $N \times N$ orthogonal matrices with determinant 1.

We want to distinguish these two cases. Thus, we consider $SO(2N)$ (even orthogonal) and $SO(2N+1)$ (odd orthogonal).

For any complex eigenvalue of an orthogonal matrix, its complex conjugate is also an eigenvalue. The eigenvalues of $X \in SO(2N)$ can be written as

$$e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N}$$

with

$$0 \leq \theta_j \leq \pi.$$

The Weyl integration formula for integrating a symmetric function $f(X) = f(\theta_1, \ldots, \theta_N)$ over $SO(2N)$ is

$$\begin{aligned} \int_{SO(2N)} f(X)\, dX &= \frac{2^{(N-1)^2}}{\pi^N N!} \int_{[0,\pi]^N} f(\theta_1, \ldots, \theta_N) \\ &\quad \times \prod_{1 \leq j < k \leq N} (\cos\theta_k - \cos\theta_j)^2 d\theta_1 \ldots d\theta_N. \end{aligned}$$

The eigenvalues of $X \in SO(2N+1)$ can be written as

$$1, e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N}$$

with

$$0 \leq \theta_j \leq \pi.$$

The Weyl integration formula for integrating a symmetric function $f(X) = f(\theta_1, \ldots, \theta_N)$ over the space $SO(2N+1)$ is

$$\int_{SO(2N+1)} f(X)\; dX \;=\; \frac{2^{N^2}}{\pi^N N!} \int_{[0,\pi]^N} f(\theta_1, \ldots, \theta_N) \prod_{1\le j<k\le N} (\cos\theta_k - \cos\theta_j)^2 \times \prod_{h=1}^{N} \sin^2 \frac{\theta_h}{2}\; d\theta_1 \ldots d\theta_N.$$

A unitary matrix X is said to be *symplectic* if $XZX^t = Z$ where

$$Z = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}.$$

A symplectic matrix necessarily has determinant equal to 1. The *symplectic group* $\mathrm{USp}(2N)$ is the subgroup of $2N \times 2N$ symplectic matrices. The eigenvalues of a symplectic matrix are

$$e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N}$$

with

$$0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_N \le \pi.$$

The Weyl integration formula for integrating a symmetric function $f(X) = f(\theta_1, \ldots, \theta_N)$ over $USp(2N)$ is

$$\int_{USp(2N)} f(X)\; dX \;=\; \frac{2^{N^2}}{\pi^N N!} \int_{[0,\pi]^N} f(\theta_1, \ldots, \theta_N) \prod_{1\le j<k\le N} (\cos\theta_k - \cos\theta_j)^2 \prod_{h=1}^{N} \sin^2 \theta_h\; d\theta_1 \ldots d\theta_N.$$

4 Vandermonde determinants and orthogonal polynomials

We occasionally use the notation $(f(j,k))_{j,k}$ to denote the matrix whose j,k entry is $f(j,k)$.

We recall the basic fact about Vandermonde determinants. For any set N-tuple of complex numbers $(x_1, \ldots, x_N)$ let

$$\Delta(x_1, \ldots, x_N) = \det_{N\times N} \left(x_k^{j-1}\right)_{jk}. \tag{2}$$

Then

$$\Delta(x_1, \ldots, x_N) = \prod_{1\le j<k\le N} (x_k - x_j). \tag{3}$$

To prove this, one observes that both sides are homogeneous polynomials of total degree $N(N-1)/2$ which vanish whenever $x_j = x_k$. This fact identifies the two sides up to a constant factor. That the coefficient of $x_N^{N-1} x_{N-1}^{N-2} \ldots x_2$ is 1 in both expressions completes the proof.

Observe that

$$\prod_{1\le j<k\le N} |e^{i\theta_k} - e^{i\theta_j}|^2 = |\Delta(e^{i\theta_1}, \ldots, e^{i\theta_N})|^2 \tag{4}$$

and

$$\prod_{1\le j<k\le N} (\cos\theta_k - \cos\theta_j)^2 = \Delta(\cos\theta_1, \ldots, \cos\theta_N)^2. \tag{5}$$

Useful in our calculations will be the

Lemma 1 (Transposing Lemma) *We have*

$$\det_{N\times N}(\phi_{j-1}(x_k)) \det_{N\times N}(\psi_{j-1}(y_k)) = \det_{N\times N}\Big(\sum_{n=1}^{N} \phi_{n-1}(x_j)\psi_{n-1}(y_k)\Big). \tag{6}$$

This identity just follows by using the fact that the determinant of a matrix and its transpose are the same, and matrix multiplication. Specifically,

$$\begin{aligned} \det_{N\times N}(\phi_{j-1}(x_k)) \det_{N\times N}(\psi_{j-1}(y_k)) &= \det_{N\times N}(\phi_{n-1}(x_j))_{j,n} \det_{N\times N}(\psi_{n-1}(y_k))_{n,k} \\ &= \det\Big(\sum_{n=1}^{N} \phi_{n-1}(x_j)\psi_{n-1}(y_k)\Big)_{j,k}. \end{aligned}$$

4.1 An alternate formula for the Haar measure on $U(N)$

In order to compute the statistics we desire, we require an alternate formula for the Haar measure. The Transposing Lemma implies the identity

$$\prod_{1\le j<k\le N} |e^{i\theta_k} - e^{i\theta_j}|^2 = \det_{N\times N}\big(S_N(\theta_k-\theta_j)\big) \tag{7}$$

where

$$S_N(\theta) = \frac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}. \tag{8}$$

From this identity we have

$$dX_{U_N} = \frac{d\theta_1\ldots d\theta_N}{(2\pi)^N N!}\det_{N\times N}\big(S_N(\theta_k-\theta_j)\big). \tag{9}$$

To prove this we apply the Transposing Lemma with $\phi_j(x_k) = e^{ij\theta_k}$ and $\psi_j(x_k) = e^{-ij\theta_k}$ and use the fact that

$$\sum_{n=1}^{N} e^{i(n-1)\theta} = \frac{e^{iN\theta}-1}{e^{i\theta}-1} = \frac{e^{iN\theta/2}}{e^{i\theta/2}}\frac{e^{iN\theta/2}-e^{-iN\theta/2}}{e^{i\theta/2}-e^{-i\theta/2}} = e^{i(N-1)\theta/2}S_N(\theta)$$

from which

$$\begin{aligned} |\det(e^{i(j-1)\theta_k})|^2 &= \det\Big(\sum_{n=1}^{N} e^{i(n-1)(\theta_j-\theta_k)}\Big)_{j,k} \\ &= \det\big(e^{iN(\theta_j-\theta_k)/2}S_N(\theta_j-\theta_k)\big) \\ &= \det\big(S_N(\theta_j-\theta_k)\big); \end{aligned}$$

the last line holds by factoring out $e^{iN\theta_j/2}$ from the jth row and $e^{-iN\theta_k/2}$ from the kth column and observing that the product of all of these factors is 1.

For future reference we introduce the notation

$$S(x) = \frac{\sin\pi x}{\pi x}. \tag{10}$$

4.2 Alternate formulas for orthogonal and symplectic Haar measures

Now we give alternate formulas for our other measures. To accomplish this, it is helpful to first recall the basic properties of the Tchebychev polynomials. Let $T_n(x)$ be the (Chebyshev) polynomial of degree n for which

$$T_n(\cos\theta) = \cos n\theta \tag{11}$$

and $U_n(x)$ is the polynomial of degree n for which

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{12}$$

Thus, $T_0(x) = 1, T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$ and so on and $U_0(x) = 1$, $U_1(x) = 2x$, $U_2(x) = 4x^2 - 1$, $U_3(x) = 8x^3 - 4x$, and so on. From $\cos(n+1)\theta = \cos n\theta\cos\theta - \sin n\theta\sin\theta$ and $\sin(n+1)\theta = \sin n\theta\cos\theta + \cos n\theta\sin\theta$ it is easy to see that

$$T_{n+1}(x) = xT_n(x) - (1-x^2)U_{n-1}(x)$$

and

$$U_n(x) = xU_{n-1}(x) + T_n(x).$$

Thus,

$$\begin{aligned}
T_{n+2}(x) &= xT_{n+1}(x) - (1-x^2)U_n(x) \\
&= xT_{n+1}(x) - (1-x^2)(xU_{n-1}(x) + T_n(x)) \\
&= xT_{n+1}(x) - (1-x^2)T_n(x) + x(T_{n+1}(x) - xT_n(x)) \\
&= 2xT_{n+1}(x) - T_n(x).
\end{aligned}$$

Similarly, $U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x)$. Notice that the leading coefficient in $T_n(x)$ is 2^{n-1} and in $U_n(x)$ it is 2^n.

Finally, we let $V_n(x)$ be the polynomial of degree n for which

$$V_n(\cos\theta) = U_{2n}\big(\cos\frac{\theta}{2}\big) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}. \tag{13}$$

It can be shown that $V_n(x) = 2^n x^n + \dots$ has leading coefficient 2^n.

Now $\prod_{1\le j<k\le N}(\cos\theta_k - \cos\theta_j) = \Delta(\cos\theta_1,\ldots,\cos\theta_N)$. Let $x_j = \cos\theta_j$ for convenience. Then, by elementary row operations, $\Delta(x_1,\ldots,x_N)$

$$
\begin{aligned}
&= \det\begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_N \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{N-2} & x_2^{N-2} & \ldots & x_N^{N-2} \\ x_1^{N-1} & x_2^{N-1} & \ldots & x_N^{N-1} \end{pmatrix} \\
&= \frac{1}{2^{N-2}}\det\begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_N \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{N-2} & x_2^{N-2} & \ldots & x_N^{N-2} \\ 2^{N-2}x_1^{N-1} & 2^{N-2}x_2^{N-1} & \ldots & 2^{N-2}x_N^{N-1} \end{pmatrix} \\
&= \frac{1}{2^{N-2}}\det\begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_N \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{N-2} & x_2^{N-2} & \ldots & x_N^{N-2} \\ T_{N-1}(x_1) & T_{N-1}(x_2) & \ldots & T_{N-1}(x_N) \end{pmatrix}
\end{aligned}
$$

by adding appropriate multiples of the first $N-1$ rows to the last row. Now we do the same thing to all of the rows, except the first which we leave alone, working our way from the bottom to the top. In this way, we find that

$$\Delta(\cos\theta_1,\ldots,\cos\theta_N) = 2^{-(N-1)(N-2)/2}\det_{N\times N}(T_{j-1}(\cos\theta_k)) \tag{14}$$

and also

$$\Delta(\cos\theta_1,\ldots,\cos\theta_N) = 2^{-(N-1)(N-2)/2-N}\det_{N\times N}(U_{j-1}(\cos\theta_k)). \tag{15}$$

For the Haar measure on SO$(2N)$, we have

$$
\begin{aligned}
dX_{SO(2N)} &= \frac{2^{(N-1)^2}}{\pi^N N!}\Delta(\cos\theta_1,\ldots,\cos\theta_N)^2 d\theta_1\ldots d\theta_N \\
&= \frac{2^{N-1}}{\pi^N N!}\det_{N\times N}\big(T_{j-1}(\cos\theta_k)\big)^2 d\theta_1\ldots d\theta_N.
\end{aligned}
$$

If we multiply each row except the first by $\sqrt{2}$ we find that

$$\Delta(\cos\theta_1,\ldots,\cos\theta_N) = 2^{-(N-1)^2/2}\det_{N\times N}(T^*_{j-1}(\cos\theta_k))$$

where we let $T_j^* = \sqrt{2}T_j$ for $j \geq 1$ and $T_0^* = T_0 = 1$. Then,

$$dX_{SO(2N)} = \frac{1}{\pi^N N!} \det_{N\times N} \big(T_{j-1}^*(\cos\theta_k)\big)^2 d\theta_1 \dots d\theta_N. \tag{16}$$

By the Transposing Lemma,

$$\Delta(\cos\theta_1,\dots,\cos\theta_N)^2 = 2^{-(N-1)^2} \det_{N\times N} \big(1 + 2\sum_{n=1}^{N-1} \cos n\theta_j \cos n\theta_k\big) \tag{17}$$

Recall that $S_N(\theta) = \frac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}$ and $S(x) = \frac{\sin \pi x}{\pi x}$. Now

$$\begin{aligned}\sum_{n=-N}^{N} \cos nx &= \Re \sum_{n=-N}^{N} e^{inx} = \Re \frac{e^{i(N+1)x} - e^{-iNx}}{e^{ix}-1} \\ &= \Re \frac{e^{i(N+1/2)x} - e^{-i(N+1/2)x}}{e^{ix/2} - e^{-ix/2}} = \frac{\sin(N+1/2)x}{\sin x/2} = S_{2N+1}(x).\end{aligned}$$

Consequently,

$$\sum_{n=1}^{N} \cos nx = \frac{S_{2N+1}(x) - 1}{2}$$

so that

$$\begin{aligned}1 + 2\sum_{n=1}^{N-1} \cos nx \cos ny &= 1 + \sum_{n=1}^{N-1} \big(\cos n(x-y) + \cos n(x+y)\big) \\ &= \frac{S_{2N-1}(x-y) + S_{2N-1}(x+y)}{2}.\end{aligned}$$

Consequently, a basic identity is

$$2^{(N-1)^2} \prod_{1\leq j<k\leq N} (\cos\theta_k - \cos\theta_j)^2 = \det_{N\times N} \big(\frac{S_{2N-1}(\theta_k-\theta_j) + S_{2N-1}(\theta_k+\theta_j)}{2}\big)$$

from which we deduce by (16) that

$$\begin{aligned}dX_{SO(2N)} &= \frac{1}{\pi^N N!} \det_{N\times N} \big(\frac{S_{2N-1}(\theta_k-\theta_j) + S_{2N-1}(\theta_k+\theta_j)}{2}\big)\ d\theta_1 \dots d\theta_N \\ &= \frac{1}{\pi^N N!} \det_{N\times N} \big(K_{SO(2N)}(\theta_j,\theta_k)\big)\ d\theta_1 \dots d\theta_N,\end{aligned} \tag{18}$$

where we define

$$K_{SO(2N)}(x,y) = \frac{S_{2N-1}(y-x) + S_{2N-1}(y+x)}{2}$$

and, for use in a moment,

$$K_{USp(2N)}(x,y) = \frac{S_{2N+1}(y-x) - S_{2N+1}(y+x)}{2}$$

and

$$K_{SO(2N+1)}(x,y) = \frac{S_{2N}(y-x) - S_{2N}(y+x)}{2}.$$

Now we do the same for the Haar measure of the symplectic group. Again, by elementary row operations on the determinant $\Delta(x_1,\dots,x_N)$, we find that (recall that the leading coefficient of the Chebyshev polynomial $U_N(x)$ is $(2x)^N$),

$$\Delta(x_1,\dots,x_N) = 2^{-N(N-1)/2} \det_{N\times N}(U_{j-1}(x_k)).$$

Then $dX_{USp(2N)}$, the Haar measure on USp$(2N)$, satisfies

$$\begin{aligned} dX_{USp(2N)} &= \frac{2^{N^2}}{\pi^N N!}\Delta(\cos\theta_1,\dots,\cos\theta_N)^2 \prod_{n=1}^{N} \sin^2\theta_n d\theta_1\dots d\theta_N \\ &= \frac{2^N}{\pi^N N!} \det_{N\times N}\big(U_{j-1}(\cos\theta_k)\big)^2 \prod_{n=1}^{N}\sin^2\theta_n d\theta_1\dots d\theta_N. \end{aligned}$$

Now, by the Transposing Lemma

$$\begin{aligned} \prod_{1\le j<k\le N}(\cos\theta_k - \cos\theta_j)^2 \prod_{n=1}^{N}\sin^2\theta_n &= 2^{-N(N-1)} \det_{N\times N}\left(\sum_{n=1}^{N}\sin n\theta_j \sin n\theta_k\right) \\ &= 2^{-N(N-1)} \det_{N\times N}\left(\frac{S_{2N+1}(\theta_k-\theta_j) - S_{2N+1}(\theta_k+\theta_j)}{2}\right) \end{aligned}$$

since

$$\begin{aligned} 2\sum_{n=1}^{N}\sin nx \sin ny &= \sum_{n=1}^{N}\big(\cos n(x-y) - \cos n(x+y)\big) \\ &= \frac{S_{2N+1}(x-y) - S_{2N+1}(x+y)}{2}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
dX_{USp(2N)} &= \frac{1}{\pi^N N!} \det_{N\times N} \Big(\frac{S_{2N+1}(\theta_k-\theta_j)-S_{2N+1}(\theta_k+\theta_j)}{2}\Big)\ d\theta_1\dots d\theta_N \\
&= \frac{1}{\pi^N N!} \det_{N\times N} \big(K_{USp(2N)}(\theta_k,\theta_j)\big)\ d\theta_1\dots d\theta_N. \qquad (19)
\end{aligned}
$$

Finally, $dX_{SO(2N+1)}$, the Haar measure on $SO(2N+1)$, satisfies

$$
\begin{aligned}
dX_{SO(2N+1)} &= \frac{2^{N^2}}{\pi^N N!}\Delta(\cos\theta_1,\dots,\cos\theta_N)^2 \prod_{n=1}^{N} \sin^2\frac{\theta_n}{2} d\theta_1\dots d\theta_N \\
&= \frac{2^N}{\pi^N N!} \det_{N\times N}\big(V_{j-1}(\cos\theta_k)\big)^2 \prod_{n=1}^{N} \sin^2\frac{\theta_n}{2} d\theta_1\dots d\theta_N.
\end{aligned}
$$

By the Transposing Lemma,

$$
\begin{aligned}
dX_{SO(2N+1)} &= \frac{2^N}{\pi^N N!} \det_{N\times N}\Big(\sum_{n=1}^{N} \sin(n-\tfrac{1}{2})\theta_j \sin(n-\tfrac{1}{2})\theta_k\Big) d\theta_1\dots d\theta_N \\
&= \frac{1}{\pi^N N!} \det_{N\times N}\Big(\frac{S_{2N}(\theta_k-\theta_j)-S_{2N}(\theta_k+\theta_j)}{2}\theta_k\Big) d\theta_1\dots d\theta_N.
\end{aligned}
$$

Therefore,

$$
dX_{SO(2N+1)} = \frac{1}{\pi^N N!} \det_{N\times N}\big(K_{SO(2N+1)}(\theta_k,\theta_j)\big)\ d\theta_1\dots d\theta_N. \qquad (20)
$$

5 Andréief's identity

As a consistency check, we now deduce that our measures have total mass one. To do this we use a formula of Andréief:

Lemma 2 (Andréief's identity) *For any interval J and integrable functions ϕ_j and ψ_j:*

$$
\frac{1}{N!}\int_{J^N} \det_{N\times N}(\phi_j(\theta_k)) \det_{N\times N}(\psi_j(\theta_k))\ d\theta_1\dots d\theta_N = \det_{N\times N}\Big(\int_J \phi_j(\theta)\psi_k(\theta)\ d\theta\Big). \quad (21)
$$

5.1 Proof of Andréief's identity

We use the definition of determinant for a matrix $X = (x_{jk})$:

$$\det X = \sum_{\sigma \in \pi_N} \operatorname{sgn}(\sigma) \prod_{j=1}^{N} x_{j,\sigma j}$$

where π_N denotes the collection of the $N!$ permutations of $[1, N] := \{1, 2, \ldots, N\}$. Thus,

$$\begin{aligned}
&\det_{N\times N} \big(\phi_j(\theta_k)\big) \det_{N\times N} \big(\psi_j(\theta_k)\big) \\
&= \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \sum_{\tau} \operatorname{sgn}(\tau) \prod_{k=1}^{N} \psi_k(\theta_{\tau k}) \qquad (22) \\
&\underset{\tau \to \sigma\tau}{=} \sum_{\sigma,\tau} \operatorname{sgn}(\tau) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \prod_{k=1}^{N} \psi_k(\theta_{\sigma\tau k}) \ \prod_{i=1}^{N} d\theta_i \\
&\underset{k \to \tau^{-1}k}{=} \sum_{\sigma,\tau} \operatorname{sgn}(\tau) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \prod_{k=1}^{N} \psi_{\tau^{-1}k}(\theta_{\sigma k}) \\
&= \sum_{\sigma,\tau} \operatorname{sgn}(\tau) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \psi_{\tau^{-1}j}(\theta_{\sigma j}) \\
&\underset{\tau \to \tau^{-1}}{=} \sum_{\sigma,\tau} \operatorname{sgn}(\tau) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \psi_{\tau j}(\theta_{\sigma j}).
\end{aligned}$$

Now if we integrate each θ_j over the interval J the total integral over J^N splits into a product of N integrals over J and we obtain the result.

Note the slightly more general result

$$\begin{aligned}
&\frac{1}{N!} \int_{J^N} \prod_{i=1}^{N} f(\theta_i) \det_{N\times N} (\phi_j(\theta_k)) \det_{N\times N} (\psi_j(\theta_k)) \ d\theta_1 \ldots d\theta_N \qquad (23) \\
&\qquad = \det_{N\times N} \Big(\int_J f(\theta) \phi_j(\theta) \psi_k(\theta) \ d\theta \Big).
\end{aligned}$$

This follows by replacing $\phi_j(\theta_k)$ on the left side of Andréief's identity by $\phi_j(\theta_k) f(\theta_k)$.

5.2 Andréief with different size determinants

Lemma 3 (Andréief's general identity) *For any interval J and integrable functions $\phi_j, 1 \le j \le N$ and $\psi_j, 1 \le j \le N+L$, we have*

$$\frac{1}{N!}\int_{J^N} \det_N(\phi_j(\theta_k)) \det_{N+L}(\psi_j(\theta_k))\ d\theta_1 \dots d\theta_N = \det_{N+L}\big(\mathcal{T}_j\phi_j(\theta)\psi_k(\theta)\big) \tag{24}$$

where

$$\mathcal{T}_j\phi_j(\theta)\psi_k(\theta) = \begin{cases} \int_J \phi_j(\theta)\psi_k(\theta)\ d\theta & \text{if } j \le N \\ \psi_k(\theta_j) & \text{if } j > N \end{cases}$$

The proof is a minor modification of the proof we already gave in the case that $L = 0$. For a permutation $\sigma \in \pi_N$ let $\sigma' \in \pi_{N+L}$ be defined by

$$\sigma'(j) = \begin{cases} \sigma(j) & \text{if } j \le N \\ j & \text{if } j > N \end{cases}$$

Further, define the functions ϕ'_j by

$$\phi'_j(\theta) = \begin{cases} \phi_j(\theta) & \text{if } j \le N \\ 1 & \text{if } j > N \end{cases}$$

Then we have

$$\begin{aligned}
&\det_N\big(\phi_j(\theta_k)\big) \det_{N+L}\big(\psi_j(\theta_k)\big) \\
&= \sum_{\sigma\in\pi_N} \operatorname{sgn}(\sigma)\prod_{j=1}^{N}\phi_j(\theta_{\sigma j}) \sum_{\tau\in\pi_{N+L}} \operatorname{sgn}(\tau)\prod_{k=1}^{N+L}\psi_k(\theta_{\tau k}) \qquad (25)\\
&\underset{\tau\,\to\,\sigma'\tau}{=} \sum_{\sigma,\tau}\operatorname{sgn}(\tau)\prod_{j=1}^{N}\phi_j(\theta_{\sigma j})\prod_{k=1}^{N+L}\psi_k(\theta_{\sigma'\tau k}) \\
&\underset{k\,\to\,\tau^{-1}k}{=} \sum_{\sigma,\tau}\operatorname{sgn}(\tau)\prod_{j=1}^{N}\phi_j(\theta_{\sigma j})\prod_{k=1}^{N+L}\psi_{\tau^{-1}k}(\theta_{\sigma' k}) \\
&= \sum_{\sigma,\tau}\operatorname{sgn}(\tau)\prod_{j=1}^{N+L}\phi'_j(\theta_{\sigma' j})\psi_{\tau^{-1}j}(\theta_{\sigma' j}) \\
&\underset{\tau\,\to\,\tau^{-1}}{=} \sum_{\sigma,\tau}\operatorname{sgn}(\tau)\prod_{j=1}^{N+L}\phi'_j(\theta_{\sigma' j})\psi_{\tau j}(\theta_{\sigma' j}).
\end{aligned}$$

Now, if we integrate each θ_j for $1 \le j \le N$ over the interval J we have

$$\begin{aligned}
&\int_{J^N} \det_N(\phi_j(\theta_k)) \det_{N+L}(\psi_j(\theta_k))\ d\theta_1 \dots d\theta_N \\
&\quad = N! \sum_\tau \operatorname{sgn}(\tau) \prod_{j=1}^{N} \int_J \phi_j(\theta)\psi_{\tau j}(\theta)\ d\theta \prod_{j=N+1}^{N+L} \psi_{\tau j}(\theta_j) \\
&\quad = N! \sum_\tau \operatorname{sgn}(\tau) \prod_{j=1}^{N+L} \mathcal{T}_j \phi_j(\theta)\psi_{\tau j}(\theta) \\
&\quad = N! \det_{N+L} \big(\mathcal{T}_j \phi_j(\theta)\psi_k(\theta)\big).
\end{aligned}$$

5.3 Verification that the Haar measures have total mass 1

Using

$$\phi_j(\theta) = e^{i(j-1)\theta}$$

we see that

$$\begin{aligned}
\int_{[0,2\pi]^N} dX_{U(N)} &= \int_{[0,2\pi]^N} |\det_{N\times N}(e^{i(j-1)\theta_k})|^2 \frac{d\theta_1 \dots\ d\theta_N}{N!(2\pi)^N} \\
&= \frac{1}{(2\pi)^N} \det_{N\times N} \big(\int_0^{2\pi} e^{i(j-1)\theta} e^{-i(k-1)\theta}\ d\theta\big) \\
&= \frac{1}{(2\pi)^N} \det_{N\times N} \big(2\pi I\big) = 1.
\end{aligned}$$

This shows that the total mass of the Haar measure of $U(N)$ is 1.

We observe further that (16) and Andréief's identity together imply that

$$\int_{[0,\pi]^N} dX_{SO(2N)} = \frac{2^{N-1}}{\pi^N} \det_{N\times N} \left(\int_0^\pi T_{j-1}(\cos\theta) T_{k-1}(\cos\theta)\ d\theta \right) = 1,$$

since

$$\begin{aligned}
\int_0^\pi T_{j-1}(\cos\theta) T_{k-1}(\cos\theta)\ d\theta &= \int_0^\pi \cos(j-1)\theta \cos(k-1)\theta\ d\theta \\
&= \frac{1}{2} \int_0^\pi \big(\cos(j+k-2)\theta + \cos(j-k)\theta\big)\ d\theta \\
&= \frac{\pi}{2} \delta_{j,k}(1 + \delta_{1,j})
\end{aligned}$$

because the integral is 0 unless $j = k$ in which case it is π if $j > 1$ and 2π if $j = 1$. Also,

$$\begin{aligned}\int_{USp(2N)} dX &= \frac{2^N}{\pi^N} \det_{N\times N}\left(\int_0^\pi \sin^2\theta\ U_{j-1}(\cos\theta)\ U_{k-1}(\cos\theta)\ d\theta\right) \\ &= \frac{2^N}{\pi^N} \det_{N\times N}\left(\int_0^\pi \sin j\theta \sin k\theta\ d\theta\right) \\ &= \frac{2^N}{\pi^N} \det_{N\times N}\left(\frac{1}{2}\int_0^\pi \big(\cos(j-k)\theta - \cos(j+k)\theta\big)\ d\theta\right).\end{aligned}$$

Since the integral is π when $j = k$ and 0 otherwise, this confirms that the total measure of $USp(2N)$ is 1.

Similarly, we can calculate that the total mass of $SO(2N+1)$ is 1.

6 Gaudin's Lemma

The following Lemma is the key to begin computing the statistics of interest.[1]

Lemma 4 (Gaudin's Lemma) *Suppose that we have a function f and a measurable set J such that*

$$\int_J f(x,\theta)f(\theta,y)\ d\theta = Cf(x,y) \tag{26}$$

for all x and y where $C = C(J,f)$ is a constant. Suppose also that

$$\int_J f(x,x)\ dx = D, \tag{27}$$

where $D = D(J,f)$ is constant. Then

$$\int_J \det_{M\times M}\big(f(\theta_j,\theta_k)\big)\ d\theta_M = (D-(M-1)C)\det_{M-1}\big(f(\theta_j,\theta_k)\big). \tag{28}$$

[1]Editors' comment: This lemma is also applied in the lectures of Y.V. Fyodorov, page **??** Section 4.

This Lemma allows us to "integrate out" variables not under consideration when computing some statistic. We apply Gaudin's Lemma with $f(\theta) = S_N(\theta)$ and $J = [0, 2\pi]$, so that $D = S_N(0) = N$. Reëxpressing S_N as a geometric series and integrating term-by-term, we find that

$$\int_0^{2\pi} S_N(\theta_j - \theta) S_N(\theta - \theta_k)\ d\theta = 2\pi S_N(\theta_k - \theta_j),$$

so that $C = 2\pi$. Thus, for example,

$$\int_{[0,2\pi]} \det_{N\times N} \big(S_N(\theta_k - \theta_j)\big)\ d\theta_N = 2\pi \det_{(N-1)\times(N-1)} \big(S_N(\theta_k - \theta_j)\big).$$

Applying the Lemma repeatedly gives

$$\int_{[0,2\pi]^{N-n}} \det_{N\times N} \big(S_N(\theta_k - \theta_j)\big)\ d\theta_{n+1} \dots\ d\theta_N$$
$$= (N-n)!(2\pi)^{N-n} \det_{n\times n} \big(S_N(\theta_k - \theta_j)\big).$$

In particular,

$$\int_{U(N)} \sum_{\substack{J\subset\{1,\dots,N\}\\ J=\{j_1,\dots,j_n\}}} f(\theta_{j_1}, \dots, \theta_{j_n}) dX_{U(N)}$$
$$= \frac{1}{(2\pi)^n n!} \int_{[0,2\pi]^n} f(\theta_1, \dots, \theta_n) \det_{n\times n} S_N(\theta_k - \theta_j)\ d\theta_1 \dots d\theta_n. \quad (29)$$

Proof of Gaudin's Lemma. Let π_M be the symmetric group on $\{1, \dots, M\}$. Then,

$$\det_{M\times M} \big(f(\theta_j, \theta_k)\big) \;=\; \sum_{\sigma\in\pi_M} \operatorname{sgn}(\sigma) \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}).$$

If $\sigma M \neq M$, then

$$\int_J \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}) d\theta_M \;=\; \prod_{\substack{j=1\\ \sigma j\neq M}}^{M-1} f(\theta_j, \theta_{\sigma j}) \int_J f(\theta_{\sigma^{-1}M}, \theta_M) f(\theta_M, \theta_{\sigma M})\ d\theta_M$$
$$= f(\theta_{\sigma^{-1}M}, \theta_{\sigma M}) \prod_{\substack{j=1\\ \sigma j\neq M}}^{M-1} f(\theta_j, \theta_{\sigma j}). \quad (30)$$

For a permutation $\sigma \in \pi_M$ with $\sigma M \neq M$ define a permutation $\sigma' \in \pi_{M-1}$ by

$$\sigma' j = \begin{cases} \sigma j & \text{if } \sigma j \neq M \\ \sigma M & \text{if } \sigma j = M \end{cases}.$$

Then (30) may be reëxpressed as

$$\int_J \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}) d\theta_M = C \prod_{j=1}^{M-1} f(\theta_j, \theta_{\sigma' j}).$$

Clearly, each permutation σ' arises from $(M-1)$ different σ. Note also that $\operatorname{sgn}(\sigma') = -\operatorname{sgn}(\sigma)$. Thus, we have

$$\begin{aligned} \int_J \sum_{\substack{\sigma\in\pi_M \\ \sigma M \neq M}} \operatorname{sgn}(\sigma) \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}) d\theta_M &= -(M-1)C \sum_{\sigma' \in \pi_{M-1}} \operatorname{sgn}(\sigma') \prod_{j=1}^{M-1} f(\theta_j, \theta_{\sigma' j}) \\ &= -(M-1)C \det_{M-1} \big(f(\theta_j, \theta_k) \big). \end{aligned}$$

Now consider the σ for which $\sigma M = M$; now let σ' be defined by $\sigma' j = \sigma j$ for $j \leq M-1$. Then, for these σ, we have

$$\begin{aligned} \int_J \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}) d\theta_M &= \prod_{j=1}^{M-1} f(\theta_j, \theta_{\sigma j}) \int_J f(\theta_m, \theta_M)\ d\theta_M \\ &= D \prod_{j=1}^{M-1} f(\theta_j, \theta_{\sigma' j}). \end{aligned}$$

These σ' have the same sign as the σ they came from. Therefore,

$$\begin{aligned} \int_J \sum_{\substack{\sigma\in\pi_M \\ \sigma M = M}} \operatorname{sgn}(\sigma) \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}) d\theta_M &= D \sum_{\sigma' \in \pi_{M-1}} \operatorname{sgn}(\sigma') \prod_{j=1}^{M-1} f(\theta_j, \theta_{\sigma' j}) \\ &= D \det_{M-1} \big(f(\theta_k, \theta_j) \big). \end{aligned}$$

Combining the two cases we obtain the Lemma.

6.1 Calculation for orthogonal and symplectic cases

Recall that

$$K_{SO(2N)}(x,y) = \frac{S_{2N-1}(x-y) + S_{2N-1}(x+y)}{2} = 1 + 2\sum_{n=1}^{N-1} \cos nx \cos ny$$

and

$$K_{USp(2N)}(x,y) = \frac{S_{2N+1}(x-y) - S_{2N+1}(x+y)}{2} = 2\sum_{n=1}^{N} \sin nx \sin ny.$$

Then, Gaudin's Lemma for these groups is expressed as

$$\int_{[0,\pi]^{N-n}} \det_{N\times N} \left(K_{G(N)}(\theta_j,\theta_k)\right) d\theta_{n+1} \ldots\, d\theta_N = (N-n)!\pi^{N-n} \det_{n\times n} \left(K_{G(N)}(\theta_j,\theta_k)\right)$$

where $G(N)$ can stand for $USp(2N), SO(2N)$, or $SO(2N+1)$. This allows us to "integrate out" variables in the orthogonal and symplectic settings.

Note, for future reference, that

$$\lim_{N\to\infty} \frac{1}{2N} S_{2N+1}(\pi x/N) = \frac{\sin \frac{(N+1/2)\pi x}{N}}{2N \sin \frac{\pi x}{2N}} = \frac{\sin \pi x}{\pi x} = S(x)$$

so that

$$K_G(x,y) := \lim_{N\to\infty} \frac{K_{G(N)}(\pi x/N, \pi y/N)}{2N} = S(y-x) \pm S(y+x).$$

To prove Gaudin's Lemma in this situation, it again suffices to prove the $n = N-1$ case, since the general case follows by a repeated application of this case:

$$\int_{[0,\pi]} \det_{N\times N} \left(K_{G(N)}(\theta_j,\theta_k)\right) d\theta_N = \pi \det_{(N-1)\times(N-1)} \left(K_{G(N)}(\theta_j,\theta_k)\right).$$

The key formulae are

$$\int_0^\pi K_{G(N)}(\theta_j,\theta) K_{G(N)}(\theta,\theta_k)\; d\theta = \pi K_{G(N)}(\theta_j,\theta_k).$$

Knowing this, the rest of the proof is the same; so we now verify these formulae. We calculate

$$\begin{aligned}
&\int_0^\pi K_{USp(2N)}(x,\theta)K_{USp(2N)}(\theta,y)\ d\theta \\
&\quad = \int_0^\pi \sum_{m=1}^N 2\sin mx \sin m\theta \sum_{n=1}^N 2\sin n\theta \sin ny\ d\theta \\
&\quad = 4\sum_{m,n=1}^N \sin mx \sin ny \int_0^\pi \sin m\theta \sin n\theta\ d\theta \\
&\quad = 2\sum_{m,n=1}^N \sin mx \sin ny \int_0^\pi (\cos(m-n)\theta - \cos(m+n)\theta)\ d\theta \\
&\quad = 2\pi \sum_{n=1}^N \sin nx \sin ny = \pi K_{USp(2N)}(x,y).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^\pi K_{SO(2N)}(x,\theta)K_{SO(2N)}(\theta,y)\ d\theta \\
&\quad = \int_0^\pi (1+2\sum_{m=1}^{N-1} \cos mx \cos m\theta)(1+2\sum_{n=1}^{N-1} \cos n\theta \cos ny)\ d\theta \\
&\quad = \pi + 4\sum_{m,n=1}^{N-1} \cos mx \cos ny \int_0^\pi \cos m\theta \cos n\theta\ d\theta \\
&\quad = \pi + 2\sum_{m,n=1}^{N-1} \cos mx \cos ny \int_0^\pi (\cos(m-n)\theta + \cos(m+n)\theta)\ d\theta \\
&\quad = \pi(1+2\sum_{n=1}^{N-1} \cos nx \cos ny) = \pi K_{SO(2N)}(x,y).
\end{aligned}$$

Similarly for $K_{SO(2N+1)}$.

7 n-level density

7.1 Unitary

We can use Gaudin's Lemma to compute an integral of the sort

$$\int_{U(N)} \sum_{j=1}^{N} f(\theta_j) dX,$$

or

$$\int_{U(N)} \sum_{1\le j<k\le N} f(\theta_j, \theta_k) dX,$$

or

$$\int_{U(N)} \sum_{1\le j_1<\cdots<j_n\le N} f(\theta_{j_1}, \ldots, \theta_{j_n}) dX.$$

With an obvious notation, we write the last integral as

$$\int_{U(N)} \sum_{\substack{B\subset[1,N]\\|B|=n}} f_B(\theta) dX$$

These are precisely the definitions of the 1-, 2-, and n-level densities. By Gaudin's Lemma, these integrals are, respectively,

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\ d\theta,$$

$$\frac{1}{2!(2\pi)^2}\int_{[0,2\pi]^2} f(\theta_1,\theta_2) \det_{2\times 2} S_N(\theta_k-\theta_j)\ d\theta_1\ d\theta_2,$$

and [2]

$$\frac{1}{n!(2\pi)^n}\int_{[0,2\pi]^n} f(\theta_1,\ldots,\theta_n) \det_{n\times n} S_N(\theta_k-\theta_j)\ d\theta_1 \ldots\ d\theta_n. \tag{31}$$

[2]Editors' comment: Up to a constant factor $f(\theta_1,\ldots,\theta_n)$ is being integrated against the quantity defined in the lectures of Y.V. Fyodorov, page **??**, Section 3, as the n-point correlation function.

7.2 Normalized eigenangles and large N limits

For a matrix $X \in U(N)$ with eigenvalues

$$e^{i\theta_1}, \ldots, e^{i\theta_N}$$

we let

$$\tilde{\theta}_j = \theta \frac{N}{2\pi} \tag{32}$$

be the normalized eigenangles. They satisfy

$$0 \le \tilde{\theta}_1 \le \cdots \le \tilde{\theta}_N < N.$$

The sequence of $\tilde{\theta}$ have mean spacing 1 and so give a way to compare statistics for different N. Thus, for the n-level density, we have (for a rapidly decaying smooth f)

$$\begin{aligned}
&\lim_{N\to\infty} \int_{U(N)} \sum_{\substack{B\subset[1,N]\\|B|=n}} f_B(\tilde{\theta}) dX \\
&= \lim_{N\to\infty} \frac{1}{n!(2\pi)^n} \int_{[0,2\pi]^n} f(\tilde{\theta}_1, \ldots, \tilde{\theta}_n) \det_{n\times n} S_N(\theta_k - \theta_j)\ d\theta_1 \ldots\ d\theta_n \\
&= \lim_{N\to\infty} \frac{1}{n!} \int_{[0,N]^n} f(x_1, \ldots, x_n) \det_{n\times n} \frac{1}{N} S_N\big(\frac{2\pi(x_k - x_j)}{N}\big)\ dx_1 \ldots\ dx_n \\
&= \frac{1}{n!} \int_{\mathbf{R}_+^n} f(x_1, \ldots, x_n) \det_{n\times n} S(x_k - x_j)\ dx_1 \ldots dx_n.
\end{aligned} \tag{33}$$

We say that $\det_n S(x_k - x_j)$ is the n-level density function for U.

7.3 Orthogonal and symplectic

We can use Gaudin's Lemma to compute

$$\int_{SO(2N)} \sum_{\substack{B\subset[1,N]\\|B|=n}} f_B(\theta) dX$$

and

$$\int_{USp(2N)} \sum_{\substack{B\subset[1,N]\\|B|=n}} f_B(\theta) dX.$$

By Weyl's integration formula and Gaudin's Lemma, these integrals are

$$\frac{1}{n!\pi^n}\int_{[0,\pi]^n} f(\theta_1,\ldots,\theta_n)\det_{n\times n}\left(K_{SO(2N)}(\theta_k,\theta_j)\right)\, d\theta_1\ldots\ d\theta_n$$

and

$$\frac{1}{n!\pi^n}\int_{[0,\pi]^n} f(\theta_1,\ldots,\theta_n)\det_{n\times n}\left(K_{USp(2N)}(\theta_k,\theta_j)\right)\, d\theta_1\ldots\ d\theta_n$$

respectively.

For eigenangles of matrices in $SO(2N)$ and $USp(2N)$ we let

$$\tilde{\theta}_j = \theta\frac{N}{\pi}$$

be the normalized eigenangles. Thus, for the n-level density, we have (for a rapidly decaying smooth f)

$$\begin{aligned}
&\lim_{N\to\infty}\int_{SO(2N)}\sum_{1\le j_1<\cdots<j_n\le N} f(\tilde{\theta}_{j_1},\ldots,\tilde{\theta}_{j_n})dX\\
&\quad= \lim_{N\to\infty}\frac{1}{n!\pi^n}\int_{[0,\pi]^n} f(\tilde{\theta}_1,\ldots,\tilde{\theta}_n)\det_{n\times n}\left(K_{SO(2N)}(\theta_k,\theta_j)\right)\, d\theta_1\ldots\ d\theta_n\\
&\quad= \frac{1}{n!}\int_{\mathbf{R}_+^n} f(\theta_1,\ldots,\theta_n)\det_{n\times n}\left(K_{SO}(\theta_k,\theta_j)\right)\, d\theta_1\ldots d\theta_n \qquad (34)
\end{aligned}$$

for the n-level density for SO even and

$$\begin{aligned}
&\lim_{N\to\infty}\int_{USp(2N)}\sum_{1\le j_1<\cdots<j_n\le N} f(\tilde{\theta}_{j_1},\ldots,\tilde{\theta}_{j_n})dX\\
&\quad= \lim_{N\to\infty}\frac{1}{n!\pi^n}\int_{[0,\pi]^n} f(\tilde{\theta}_1,\ldots,\tilde{\theta}_n)\det_{n\times n}\left(K_{USp(2N)}(\theta_k,\theta_j)\right)\, d\theta_1\ldots\ d\theta_n\\
&\quad= \frac{1}{n!}\int_{\mathbf{R}_+^n} f(\theta_1,\ldots,\theta_n)\det_{n\times n}\left(K_{USp}(\theta_k,\theta_j)\right)\, d\theta_1\ldots d\theta_n \qquad (35)
\end{aligned}$$

for the n-level density for USp. The n-level density for $SO(2N+1)$ is slightly complicated by the fact that the matrices in this ensemble always have an eigenangle equal to 0. This fact leads to the presence of a δ-function in the formulation of the n-level density function.

8 Correlations

8.1 Pair correlation for $U(N)$

Let f be a suitable test function and consider

$$Q_N(f) := \int_{U(N)} \sum_{j<k} f(\tilde{\theta}_j - \tilde{\theta}_k) dX.$$

Applying Gaudin's Lemma we find that

$$Q_N(f) = \int_{[0,2\pi]^2} f(\tilde{\theta}_1 - \tilde{\theta}_2) \det \begin{pmatrix} N & S_N(\theta_1 - \theta_2) \\ S_N(\theta_1 - \theta_2) & N \end{pmatrix} \frac{d\theta_1 \, d\theta_2}{2(2\pi)^2}.$$

After a change of variables, this is

$$= \frac{1}{2} \int_{[0,N]^2} f(\theta_1 - \theta_2) \det_{2\times 2} \frac{1}{N} S_N\big(\frac{2\pi(\theta_k - \theta_j)}{N}\big) \; d\theta_1 \; d\theta_2.$$

After expanding the determinant and performing another change of variables, we have

$$Q_N(f) = \frac{1}{2} \int_{[-N,N]} f(v) \left(1 - \left(\frac{S_N(2\pi v/N)}{N}\right)^2\right) (N - |v|) \; dv.$$

Now

$$\lim_{N\to\infty} \frac{1}{N} S_N\big(\frac{2\pi v}{N}\big) = \lim_{N\to\infty} \frac{1}{N} \frac{\sin \pi v}{\sin \frac{\pi v}{N}} = \frac{\sin \pi v}{\pi v} = S(v).$$

Now it follows, with a little bit of analysis, that

$$\lim_{N\to\infty} \frac{1}{N} Q_N(f) = \frac{1}{2} \int_{-\infty}^{\infty} f(v) \left(1 - \big(\frac{\sin \pi v}{\pi v}\big)^2\right) \; dv.$$

This is the same as the pair correlation for zeros of $\zeta(s)$ found by Montgomery[3]. (Note that the factor $\frac{1}{2}$ in front of our formula is because we defined out correlation sum to be over $j < k$ rather than $j \neq k$.) This important fact was fortuitously discovered at tea at the Institute for Advanced Study one afternoon in 1971 when Chowla introduced Hugh Montgomery and Freeman Dyson to each other.

[3]Editors' comment: See lectures by D.A. Goldston, page **??**, equation 6.7.

8.2 n-correlation for $U(N)$

Let $f(\theta_1,\dots,\theta_n)$ be a test-function which is translation invariant i.e.$f(\theta_1+t,\dots,\theta_n+t)=f(\theta_1,\dots,\theta_n)$. Let's suppose, for convenience, that $f(0,\theta_2,\dots,\theta_n)$ is compactly supported, say on $[0,A]$. We seek to evaluate

$$Q_N(f)=\int_{U(N)}\sum_{1\le j_1<j_2<\cdots<j_n\le N} f(\tilde\theta_{j_1},\dots,\tilde\theta_{j_n})dX.$$

By Gaudin's Lemma and a change of variables, this is

$$\begin{aligned}&=\frac{1}{n!}\int_{[0,N]^n} f(\theta_1,\dots,\theta_n)\det_{n\times n}\frac{1}{N}S_N(2\pi(\theta_j-\theta_k)/N)d\theta_1\dots d\theta_n\\ &=\int_{0\le\theta_1\le\cdots\le\theta_n\le N} f(\theta_1,\dots,\theta_n)\det_{n\times n}\frac{1}{N}S_N(2\pi(\theta_j-\theta_k)/N)d\theta_1\dots d\theta_n.\end{aligned}$$

We make the change of variable $x_1=\theta_1$, $x_2=\theta_2-\theta_1$, ... $x_n=\theta_n-\theta_1$ and the integral becomes

$$\int_0^N\int_{0\le x_2\le\cdots\le x_n\le N-x_1} g(x_1,x_2+x_1,\dots,x_n+x_1)dx_2\dots dx_n\,dx_1$$

where $g(x_1,\dots,x_n)=f(x_1,\dots,x_n)\det(\frac{S_N(2\pi(x_k-x_j)/N)}{N})$. Since g is translation invariant and $g(0,x_2,\dots,x_n)$ is compactly supported, for sufficiently large N this is

$$\begin{aligned}&=\int_0^N\int_{0\le x_2\le\cdots\le x_n\le N-x_1} g(0,x_2,\dots,x_n)dx_2\dots dx_n\,dx_1\\ &=\int_{0\le x_2\le\cdots\le x_n\le A} g(0,x_2,\dots,x_n)\int_0^{N-A}dx_1\,dx_2\dots dx_n.\end{aligned}$$

Thus

$$\lim_{N\to\infty}\frac{Q_N(f)}{N}= \frac{1}{(n-1)!}\int_{\mathbf{R}_+^{n-1}} f(x_1,x_2,\dots,x_n)\det_{n\times n}S(x_j-x_k)\big|_{x_1=0}dx_2\dots dx_n. \tag{36}$$

8.3 n-correlation for orthogonal and symplectic

Let $f(\theta_1, \ldots, \theta_n)$ be a test-function as in the last section. We seek to evaluate

$$Q_N(f) = \int_{USp(N)} \sum_{1 \le j_1 < j_2 < \cdots < j_n \le N} f(\tilde{\theta}_{j_1}, \ldots, \tilde{\theta}_{j_n}) dX.$$

By Gaudin's Lemma and a change of variables, this is

$$\begin{aligned} &= \frac{1}{n!} \int_{[0,N]^n} f(\theta_1, \ldots, \theta_n) \det_{n \times n} \frac{1}{2N} \Big(S_{2N-1}(\pi(\theta_k - \theta_j)/N) \\ &\qquad + S_{2N-1}(\pi(\theta_k + \theta_j)/N) \Big) d\theta_1 \ldots d\theta_n \\ &= \int_{0 \le \theta_1 \le \cdots \le \theta_n \le N} f(\theta_1, \ldots, \theta_n) \det_{n \times n} \frac{1}{2N} \Big(S_{2N-1}(\pi(\theta_k - \theta_j)/N) \\ &\qquad + S_{2N-1}(\pi(\theta_k + \theta_j)/N) \Big) d\theta_1 \ldots d\theta_n. \end{aligned}$$

We make the change of variable $x_1 = \theta_1$, $x_2 = \theta_2 - \theta_1$, ... $x_n = \theta_n - \theta_1$ and the integral becomes, for sufficiently large N,

$$\begin{aligned} &\int_{0 \le x_2 \le \cdots \le x_n \le A} f(0, x_2, \ldots, x_n) \int_0^{N - x_n} \det_{n \times n} \frac{1}{2N} \Big(S_{2N-1}(\pi(x_k - x_j)/N)\big|_{x_1=0} \\ &\qquad + S_{2N-1}(\pi(x_k^* + x_j^* + 2x_1)/N) \Big) dx_1 \, dx_2 \ldots dx_n \end{aligned}$$

where $x_j^* = x_j$ if $j \ne 1$ whereas $x_1^* = 0$. Now we claim that

$$\begin{aligned} &\lim_{N \to \infty} \frac{1}{N} \int_0^{N - x_n} \det_{n \times n} \frac{1}{2N} \big(S_{2N-1}(\pi(x_k - x_j)/N)\big|_{x_1=0} \\ &\qquad + S_{2N-1}(\pi(x_k^* + x_j^* + 2x_1)/N) \big) dx_1 \\ &= \det_{n \times n} S(x_k - x_j)\big|_{x_1=0}. \end{aligned}$$

To see this claim, note that in the expansion of the determinant there are $n!$ terms each of which is a product of n factors $\frac{1}{2N} S_{2N-1}(\pi(x_k - x_j))|_{x_1=0} + \frac{1}{2N} S_{2N-1}(\pi(x_k^* + x_j^* + 2x_1))$. If we multiply out each term, there are 2^n terms, all but one of which will contain at least one factor with $\frac{1}{2N} S_{2N-1}(\pi(x_k +$

$x_j + 2x_1)$). Any of the terms with at least one factor like this will tend to 0 after integrating with respect to x_1 and dividing by N; for letting

$$c(a,b,N)(x) = \frac{\sin(ax+b)}{N\sin(ax/N + b/N)},$$

it is not difficult to see that $c(a,b,N)(x) \le \frac{2}{\pi}\frac{\sin(ax+b)}{ax+b}$ provided that $ax+b < \frac{\pi N}{2}$, and $|c(a,b,N)(x)| \le 1$ for all x and integer N. Therefore, using the fact that $\int_0^B \left(\frac{\sin x}{x}\right)^j dx$ is uniformly bounded in B for each fixed j, we see that

$$\frac{1}{N}\int_0^{N-B}\prod_{j=1}^{J} c(a_j,b_j,N)(x)\; dx \to 0$$

as $N \to \infty$ through integers. This leaves only the term with all $\frac{1}{2N}S_{2N-1}(\pi(x_k - x_j))$ factors which tend to $S(x_k - x_j)$ as $N \to \infty$.

Thus, just as in the case of $U(N)$, we find that

$$\lim_{N\to\infty}\frac{Q_N(f)}{N} = \frac{1}{(n-1)!}\int_{\mathbf{R}_+^{n-1}} f(x_1,x_2,\ldots,x_n)\det_{n\times n} S(x_j - x_k)\big|_{x_1=0}dx_2\ldots dx_n.$$

In particular, the scaled limit of the n-correlation functions are the same for all of unitary, orthogonal, and symplectic groups.

9 Neighbor spacings

9.1 Nearest neighbor for $U(N)$

We derive the combinatorial relation between nearest neighbor spacings and n-correlations (see [KS], Lemma 2.3.8). For a sequence $Y : \theta_1 \le \cdots \le \theta_N$ let

$$\mathcal{S}_n(s,Y) := \#\{j : \theta_{j+n} - \theta_j \le s\},$$

and

$$\mathcal{C}_m(s,Y) := \#\{B \subset \{1,\ldots,N\} : |B| = m, \max_{j,k\in B}|\theta_j - \theta_k| \le s\}.$$

These are related to the Sep and Clump functions used in Katz-Sarnak.

Lemma 5 (Combinatorial Lemma) *For any Y,*

$$\mathcal{C}_{m+2}(s,Y) = \sum_{n\geq m} \binom{n}{m} \mathcal{S}_{n+1}(s,Y). \tag{37}$$

Proof. Take an $m+2$-tuple of indices $i_0 < i_1 < \cdots < i_{m+1}$ whose endpoints satisfy $\theta_{i_{m+1}} - \theta_{i_0} \leq s$. Let $n = i_{m+1} - i_0$ so that the pair of endpoints is counted in $\mathcal{S}_n(s,Y)$. Then there are $\binom{n-1}{m}$ sets of points of size m between these endpoints, which, taken with the endpoints can be counted in $\mathcal{C}_{m+2}(s,Y)$. Therefore, $\mathcal{C}_{m+2} = \sum \binom{n-1}{m} \mathcal{S}_n$. Adjusting the index n by one gives the result.

In general, the relation $a_m = \sum_{n\geq m} \binom{n}{m} b_n$ can be inverted to give

$$b_m = \sum_{n\geq m} (-1)^{n-m} \binom{n}{m} a_n.$$

This follows from the identity for binomial coefficients

$$\sum_{\ell=m}^{n} (-1)^{\ell} \binom{\ell}{m} \binom{n}{\ell} = \begin{cases} (-1)^m \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases}.$$

Thus, with $a_m = \mathcal{C}_{m+2}$ and $b_n = \mathcal{S}_{n+1}$ we have

Corollary 1

$$\mathcal{S}_{m+1}(s,Y) = \sum_{n\geq m} (-1)^{n-m} \binom{n}{m} \mathcal{C}_{n+2}(s,Y) \tag{38}$$

or, after adjusting the indices,

$$\mathcal{S}_m(s,Y) = \sum_{n\geq m} (-1)^{n-m-1} \binom{n-2}{m-1} \mathcal{C}_n(s,Y). \tag{39}$$

Let $\tilde{Y}_X$ be the sequence of normalized eigenangles of X. We want to compute

$$\begin{aligned} \int_0^s \mu_1(x)\,dx : &= \text{Prob}\,\{\text{Neighboring eigenangles are} < s \text{ apart}\} \\ &= \lim_{N\to\infty} \frac{1}{N} \int_{U(N)} \mathcal{S}_1(s,\tilde{Y}_X)\,dX \end{aligned}$$

and more generally

$$\int_0^s \mu_m(x)\ dx := \text{Prob}\ \{m\text{th neighboring eigenangles are} < s \text{ apart}\} = \lim_{N\to\infty} \frac{1}{N}\int_{U(N)} \mathcal{S}_m(s,\tilde{Y_X})dX.$$

Applying the n-correlation calculation with the translation invariant function

$$f(\theta_1,\ldots,\theta_n) = \prod_{1\le j<k\le n} \chi_{[0,s]}(|\theta_j-\theta_k|)$$

gives,

$$\lim_{N\to\infty}\frac{1}{N}\int_{U(N)} \mathcal{C}_n(s,\tilde{Y_X})dX = \lim_{N\to\infty}\frac{1}{N}\int_{U(N)} \sum_{\substack{B\subset\{1,\ldots,N\}\\ |B|=n}} f(\theta_B)dX$$
$$= \frac{1}{(n-1)!}\int_{[0,s]^{n-1}} \det_{n\times n} S(x_j-x_k)\big|_{x_1=0} dx_2\ldots dx_n.$$

Thus, by (38)

$$\int_0^s \mu_m(x)\ dx = \sum_{n=m+1}^{\infty} \frac{(-1)^{n-m-1}}{(n-1)!}\binom{n-2}{m-1} \times \int_{[0,s]^{n-1}} \det_{n\times n} S(x_j-x_k)\big|_{x_1=0} dx_2\ldots dx_n.$$

In particular, for the nearest neighbor spacing, we have

$$\mu_1(s) = \frac{d}{ds}\sum_{n=2}^{\infty}\frac{(-1)^n}{(n-1)!}\int_{[0,s]^{n-1}} \det_{n\times n} S(x_j-x_k)\big|_{x_1=0} dx_2\ldots dx_n.$$

Now, for any symmetric, even, translation invariant function g,

$$\frac{d}{ds}\int_{[0,s]^m} g(x_1,\ldots,x_m)dx_1\ldots dx_m = m\int_{[0,s]^{m-1}} g(s,x_2,\ldots,x_m)dx_2\ldots dx_m.$$

Therefore,

$$\begin{aligned}\mu_1(s) &= \frac{d^2}{ds^2}\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}\int_{[0,s]^n}\det_{n\times n} S(x_j-x_k)\ dx_1dx_2\ldots dx_n \\ &= \frac{d^2}{ds^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{[0,s]^n}\det_{n\times n} S(x_j-x_k)\ dx_1dx_2\ldots dx_n.\end{aligned}$$

Also, temporarily letting

$$F(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{[0,s]^n}\det_{n\times n} S(x_j-x_k)\ dx_1dx_2\ldots dx_n,$$

we have

$$\begin{aligned}\mu_m(s) &= \frac{d^2}{ds^2}\frac{d^{m-1}}{dz^{m-1}}\left(\frac{F(z)-1-z\int_0^s \det_{1\times 1} S\ d\theta}{z^2}\right)\Bigg|_{z=-1} \\ &= \frac{d^2}{ds^2}\frac{d^{m-1}}{dz^{m-1}}\frac{F(z)}{z^2}\Bigg|_{z=-1}. \end{aligned} \tag{40}$$

In the next few sections we will work toward relating the right side of this formula to another simple function.

9.2 Gram's identity

An identity of Gram is helpful for our further considerations.

Lemma 6 (Gram's Identity) *For an interval J and integrable functions ϕ_j and ψ_j,*

$$\det_{N\times N}(I+z\int_J \phi_j(x)\psi_k(x)\ dx)=\sum_{n=0}^{N}\frac{z^n}{n!}\int_{J^n}\left(\det_{n\times n}\sum_{h=1}^{N}\phi_h(x_j)\psi_h(x_k)\right)\ dx_1\ldots dx_n. \tag{41}$$

Proof . The left-hand-side of (41) is

$$
\begin{aligned}
&\det_{N\times N}(I + z\int_J \phi_j(x)\psi_k(x)\ dx)\\
&\quad = \sum_{\sigma\in\pi_N} \operatorname{sgn}(\sigma) \prod_{j=1}^{N} \big(\delta_{j,\sigma j} + z\int_J \phi_j(x)\psi_{\sigma j}(x)\ dx\big)\\
&\quad = \sum_{\sigma\in\pi_N} \operatorname{sgn}(\sigma) \sum_{A\subset N} \prod_{j\notin U} \delta_{j,\sigma j} \prod_{j\in U} z\int_J \phi_j(x)\psi_{\sigma j}(x)\ dx\\
&\quad = \sum_{A\subset N} z^{|A|} \sum_{\sigma\in\pi_A} \operatorname{sgn}(\sigma) \prod_{j\in A} \int_J \phi_j(x)\psi_{\sigma j}(x)\ dx\\
&\quad = \sum_{n=0}^{N} z^n \sum_{\substack{A\subset N\\ |A|=n}} \det_A \int_J \phi_j(x)\psi_k(x)\ dx
\end{aligned}
$$

and the right-hand-side is

$$
\begin{aligned}
= \quad & \sum_{n=0}^{N} \frac{z^n}{n!} \int_{J^n} \sum_{\sigma\in\pi_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} \sum_{h=1}^{N} \phi_h(x_j)\psi_h(x_{\sigma j})\ dx_1 \ldots dx_N\\
= \quad & \sum_{n=0}^{N} \frac{z^n}{n!} \int_{J^n} \sum_{\sigma\in\pi_n} \operatorname{sgn}(\sigma) \sum_{\lambda:[1,n]\to[1,N]} \prod_{j=1}^{n} \phi_{\lambda j}(x_j)\psi_{\lambda j}(x_{\sigma j})\ dx_1 \ldots dx_N\\
= \quad & \sum_{n=0}^{N} \frac{z^n}{n!} \sum_{\sigma\in\pi_n} \operatorname{sgn}(\sigma) \sum_{\lambda:[1,n]\to[1,N]} \prod_{j=1}^{n} \int_J \phi_{\lambda j}(x)\psi_{\lambda\sigma^{-1}j}(x)\ dx\\
= \quad & \sum_{n=0}^{N} \frac{z^n}{n!} \sum_{\lambda:[1,n]\to[1,N]} \sum_{\sigma\in\pi_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} \int_J \phi_{\lambda j}(x)\psi_{\lambda\sigma^{-1}j}(x)\ dx\\
= \quad & \sum_{n=0}^{N} \frac{z^n}{n!} \sum_{\lambda:[1,n]\to[1,N]} \det\big(\int_J \phi_{\lambda j}(x)\psi_{\lambda k}(x)\ dx\big).
\end{aligned}
$$

If λ is not one-to-one, then the inner determinant is 0. If λ is one-to-one, call the image A. Each such set A appears $n!$ times and we get the left-hand-side.

Remark. This proof is reminiscent of the proof that that the determinant of a product is the product of the determinants. Thus,

$$\begin{aligned} \det_{N\times N}(AB) &= \det(a_{jh})(b_{hk}) = \det(\sum_{h=1}^{N} a_{jh}b_{hk}) \\ &= \sum_{\sigma\in\pi_N} \mathrm{sgn}(\sigma) \prod_{j=1}^{N}\sum_{h=1}^{N} a_{jh}b_{h,\sigma j} \\ &= \sum_{\sigma\in\pi_N} \mathrm{sgn}(\sigma) \sum_{\lambda:[1,N]\to[1,N]} \prod_{j=1}^{N} a_{j,\lambda j}b_{\lambda j,\sigma j} \end{aligned}$$

where the sum over λ is over all of the N^N functions from $[1, N]$ to itself. Now, we claim that for each fixed λ the sum

$$\sum_{\sigma\in\pi_N} \mathrm{sgn}(\sigma) \prod_{j=1}^{N} a_{j,\lambda j}b_{\lambda j,\sigma j}$$

is 0 unless λ is actually a permutation. For suppose that $\lambda u = \lambda v$ for some $u \neq v \in [1, N]$. Then, for each σ let σ' be the permutation defined by $\sigma' j = \sigma j$ if $j \neq u, v$ whereas

$$\sigma' j = \begin{cases} \sigma u \text{ if } j = v \\ \sigma v \text{ if } j = u \end{cases}.$$

In this way π_N splits up into pairs (σ, σ') of permutations. Note that $\mathrm{sgn}(\sigma) = -\mathrm{sgn}(\sigma')$. Then

$$\begin{aligned} \prod_{j=1}^{N} a_{j,\lambda j}b_{\lambda j,\sigma j} &= a_{u,\lambda u}b_{\lambda u,\sigma u}a_{v,\lambda v}b_{\lambda v,\sigma v} \prod_{j\neq u,v} a_{j,\lambda j}b_{\lambda j,\sigma j} \\ &= a_{u,\lambda u}b_{\lambda v,\sigma' v}a_{v,\lambda v}b_{\lambda u,\sigma' u} \prod_{j\neq u,v} a_{j,\lambda j}b_{\lambda j,\sigma' j} \end{aligned}$$

.

Thus, the contribution from σ cancels that from σ' in the case that $\lambda u = \lambda v$.

Therefore,

$$\begin{aligned}
\det(AB) &= \sum_{\sigma,\tau\in\pi_N} \operatorname{sgn}(\sigma) \prod_{j=1}^{N} a_{j,\tau j} b_{\tau j,\sigma j} \\
&= \sum_{\sigma,\tau\in\pi_N} \operatorname{sgn}(\sigma) \prod_{j=1}^{N} a_{j,\tau j} \prod_{k=1}^{N} b_{\tau k,\sigma k} \\
&\underset{\sigma\to\sigma\tau}{=} \sum_{\sigma,\tau\in\pi_N} \operatorname{sgn}(\sigma\tau) \prod_{j=1}^{N} a_{j,\tau j} \prod_{k=1}^{N} b_{\tau k,\sigma\tau k} \\
&\underset{k\to\tau^{-1}k}{=} \sum_{\sigma,\tau\in\pi_N} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \prod_{j=1}^{N} a_{j,\tau j} \prod_{k=1}^{N} b_{k,\sigma k} \\
&= \det A \det B.
\end{aligned}$$

9.3 Intervals with precisely n eigenvalues

Let $E_{G(N)}(n, J)$ be the measure of the set of matrices $A \in G(N)$ which have precisely n eigenvalues in the interval J. Here $G(N)$ can be $U(N)$, $SO(2N)$, $SO(2N+1)$, or $USp(2N)$; we denote the Haar measure by dX. Then we have a series of identities related to $E_{G(N)}(n, J)$ which will provide a basis for obtaining tractable expressions for our functions μ_m, and later for ν_j (to be introduced in the near future). Let χ_J be the characteristic function of the interval J. First of all,

$$\sum_{n=0}^{N} (1+z)^n E_{G(N)}(n, J) = \int_{G(N)} \prod_{j=1}^{N} (1 + z\chi_J(\theta_j))dX \tag{42}$$

since for any $X \in G(N)$ which has precisely n eigenvalues in J, the integrand is $(1+z)^n$. Expanding out the product on the right side gives

$$\int_{G(N)} \prod_{j=1}^{N} (1 + z\chi_J(\theta_j))dX = \sum_{n=0}^{N} z^n \binom{N}{n} \int_{G(N)} \prod_{j=1}^{n} \chi_J(\theta_j)dX. \tag{43}$$

Next by Gaudin's Lemma, (29)

$$\binom{N}{n} \int_{G(N)} \prod_{j=1}^{n} \chi_J(\theta_j)dX = \frac{1}{n!} \int_{J^n} \det_{n\times n} K_{G(N)}(\theta_j, \theta_k)\, d\theta_1 \ldots d\theta_n$$

where $K_{G(N)}(x,y)$ is the appropriate kernel for the group $G(N)$. Thus,

$$\sum_{n=0}^{N} z^n \binom{N}{n} \int_{G(N)} \prod_{j=1}^{n} \chi_J(\theta_j) dX = \sum_{n=0}^{N} \frac{z^n}{n!} \int_{J^n} \det_{n\times n} K_{G(N)}(\theta_j,\theta_k)\ d\theta_1 \ldots d\theta_n. \tag{44}$$

Now, for each $G(N)$ we can express

$$K_{G(N)}(x,y) = \sum_{h=1}^{N} \phi_{h,G}(x)\psi_{h,G}(y) \tag{45}$$

for appropriate ϕ and ψ. Therefore, by Gram's identity

$$\begin{aligned} \sum_{n=0}^{N} \frac{z^n}{n!} \int_{J^n} \det_{n\times n} K_{G(N)}(\theta_j,\theta_k)\ d\theta_1 \ldots d\theta_n \\ = \det_{N\times N} \left(I + z \int_J \phi_{j,G}(\theta)\psi_{k,G}(\theta)\ d\theta \right). \end{aligned} \tag{46}$$

Let $M_{J,G(N)}$ denote the $N \times N$ matrix with entries

$$m_{j,k} = \int_J \phi_{j,G}(\theta)\psi_{k,G}(\theta)\ d\theta. \tag{47}$$

Then

$$\det_{N\times N} \left(I + z \int_J \phi_{j,G}(\theta)\psi_{k,G}(\theta)\ d\theta \right) = \prod_{j=1}^{N} \left(1 + z\lambda_{j,G(N)}(J)\right) \tag{48}$$

where the $\lambda_{j,G(N)}(J)$ are the eigenvalues of $M_{J,G(N)}$.

We claim that if the kernel is symmetric (i.e. $K_{G(N)}(x,y) = K_{G(N)}(y,x)$), then the eigenvalues of $M_{J,G(N)}$ are also the eigenvalues of the integral operator $K_{J,G(N)}$ defined by

$$(K_{J,G(N)}f)(\theta) \quad = \quad \int_J K_{G(N)}(\theta,\mu) f(\mu)\ d\mu \tag{49}$$

acting on the N-dimensional space generated by $\{\psi_j(x) : 1 \le j \le N\}$.

Proof. Suppose that λ is an eigenvalue of $M_{J,G(N)}$ corresponding to an eigenvector $\vec{v} = (b_1, \ldots, b_N)'$ where the prime indicates transpose. Then, for each j,

$$\lambda b_j = \sum_{k=1}^{N} m_{jk} b_k = \sum_{k=1}^{N} b_k \int_J \phi_j(\theta)\psi_k(\theta)\ d\theta$$

for each j. Multiplying both sides by $\psi_j(\mu)$ and summing over j, we obtain

$$\begin{aligned} \lambda \sum_{j=1}^{N} b_j \psi_j(\mu) &= \int_J \big(\sum_{j=1}^{N} \phi_j(\theta)\psi_j(\mu)\big)\big(\sum_{k=1}^{N} b_k \psi_k(\theta)\big)\ d\theta \\ &= \int_J K_{G(N)}(\theta,\mu)\big(\sum_{k=0}^{N} b_k \psi_k(\theta)\big)\ d\theta \\ &= \int_J K_{G(N)}(\mu,\theta)\big(\sum_{k=1}^{N} b_k \psi_k(\theta)\big)\ d\theta = K_{J,G(N)} \sum_{k=1}^{N} b_k \psi_k(\mu) \end{aligned}$$

so that λ is an eigenvalue of $K_{J,G(N)}$ corresponding to the eigenfunction $f(\mu) = \sum_{k=1}^{N} b_k \psi_k(\mu)$.

Recapitulating, we have found that

$$\begin{aligned} \sum_{n=0}^{N} (1+z)^n E_{G(N)}(n,J) &= \sum_{n=0}^{N} \frac{z^n}{n!} \int_{J^n} \det_{n\times n} K_{G(N)}(\theta_j,\theta_k)\ d\theta_1 \ldots d\theta_n \\ &= \prod_{j=1}^{N} \big(1 + z\lambda_{j,G(N)}(J)\big) \end{aligned} \tag{50}$$

where the $\lambda_{j,G(N)}(J)$ are the eigenvalues of the integral operator $K_{J,G(N)}$ defined by

$$(K_{J,G(N)} f)(\theta) = \int_J K_{G(N)}(\theta,\mu) f(\mu)\ d\mu.$$

It can be shown that this equation scales appropriately for each G so that the large N limit can be taken. This results in (with an obvious notation

E_G)

$$\sum_{n=0}^{\infty}(1+z)^n E_G(n,J) = \sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{J^n}\det_{n\times n} K_G(\theta_j,\theta_k)\, d\theta_1\ldots d\theta_n \quad (51)$$
$$= \prod_{j=1}^{\infty}\big(1+z\lambda_{j,G}(J)\big)$$

where the $\lambda_{j,G}(J)$ are the eigenvalues of the integral operator $K_{J,G}$ defined by

$$(K_{J,G}f)(\theta) = \int_J K_G(\theta,\mu)f(\mu)\, d\mu.$$

The function $F(z)$ of (40) is equal to each of the above with G=U. Thus, we find for $\mu_m(s)$ that

$$\mu_m(s) = \frac{d^2}{ds^2}\frac{d^{m-1}}{dz^{m-1}}\left(z^{-2}\prod_{j=1}^{\infty}(1+z\lambda_{j,U}([0,s]))\right)\Bigg|_{z=-1}. \quad (52)$$

9.4 jth lowest eigenvalue

Let

$$\nu_{G(N)}(j,s)$$

be the density function for the jth lowest eigenvalue so that

$$\text{meas}\{A\in G(N):\ \text{the } j\text{th eigenvalue } \theta_j \text{ is smaller than } s\} \quad (53)$$
$$= \int_0^s \nu_{G(N)}(j,x)\, dx.$$

Then the set of $A\in G(N)$ with $\theta_j > s$ is the disjoint union of the set of A with exactly n eigenangles in $[0,s]$ for $n=0,1,\ldots,j-1$. Thus,

$$\int_s^{\infty}\nu_{G(N)}(j,x)dx = \sum_{n=0}^{j-1}E_{G(N)}(n,[0,s]).$$

Therefore, by (50), we have

$$\nu_{G(N)}(j,s) = -\frac{d}{ds}\sum_{n=0}^{j-1}\frac{d^n}{dz^n}\prod_{n=1}^{N}(1+z\lambda_{G(N),n}([0,s]))\big|_{z=-1}. \quad (54)$$

In the large N limit, this becomes

$$\nu_G(j,s) = -\frac{d}{ds}\sum_{n=0}^{j-1}\frac{d^n}{dz^n}\prod_{n=1}^{\infty}(1+z\lambda_{G,n}([0,s]))\Big|_{z=-1}. \tag{55}$$

For example,

$$\nu_G(1,s) = -\frac{d}{ds}\prod_{n=1}^{\infty}(1-\lambda_{G,n}([0,s]). \tag{56}$$

9.5 Relations between the eigenvalues

In this section, we develop a relationship between the eigenvalues λ_U and the eigenvalues λ_O and λ_S. (47). In the case that $J = [-s,s]$, note that if $\psi(\theta)$ is an eigenfunction of $M_{[-s,s],U(N)}$ with eigenvalue λ then $\psi(-\theta)$ is also an eigenfunction with eigenvalue λ, since

$$\lambda\psi(\theta) = \int_{-s}^{s} S_N(\theta-\mu)\psi(\mu)\ d\mu$$

implies that

$$\begin{aligned}\lambda\psi(-\theta) &= \int_{-s}^{s} S_N(-\theta-\mu)\psi(\mu)\ d\mu \\ &= \int_{-s}^{s} S_N(\theta+\mu)\psi(\mu)\ d\mu \\ &= \int_{-s}^{s} S_N(\theta-\mu)\psi(-\mu)\ d\mu.\end{aligned}$$

Therefore, if $\psi(\theta)+\psi(-\theta)\neq 0$, then it is also an eigenfunction with eigenvalue λ. A similar comment holds for $\psi(\theta)-\psi(-\theta)$. Consequently, each eigenfunction can be taken to be even or odd. The even eigenfunctions are also eigenfunctions of the integral equation with kernel

$$\frac{S_N(\mu-\theta)+S_N(\mu+\theta)}{2}$$

and the odd eigenfunctions are also eigenfunctions of the integral equation with kernel

$$\frac{S_N(\mu-\theta)-S_N(\mu+\theta)}{2}.$$

In general, if a matrix b is a "checkerboard" matrix, then the determinant of b factors. Specifically, if $b_{j,k} = 0$ whenever $i + j$ is odd, then

$$\det_{N\times N}(b_{j,k}) = \det_{[(N+1)/2]}(b_{2i-1,2j-1}) \det_{[N/2]\times[N/2]}(b_{2i,2j})$$

where $[x]$ is the greatest integer less than or equal to x.

We have such a factorization for $\det(I - M_{[-s,s],U(N)})$. Using the fact that

$$\sum_h (\delta_{jh} - \cos(j\theta)\cos(h\theta))(\delta_{hk} - \sin(h\theta)\sin(k\theta)) = \delta_{jk} - \cos(k-j)\theta$$

we deduce from (47) (see also Mehta (10.2.6)) that

$$\det(I - M_{[-s,s],U(N)}) = \det(I - M_{[-s,s],SO(2N)}) \det(I - M_{[-s,s],USp(2N)}).$$

This gives a factorization

$$\prod_{n=1}^{2N}(1 - \lambda_{n,U(2N)}(s)) = \prod_{n=1}^{N}(1 - \lambda_{n,SO(2N)}(s))(1 - \lambda_{n,USp(2N)}(s)) \tag{57}$$

into even and odd eigenvalues. In particular, in the limit we have

$$\prod_{n=1}^{\infty}(1 - \lambda_{n,U}(s)) = \prod_{n=1}^{\infty}(1 - \lambda_{n,Sp}(s))(1 - \lambda_{n,SO,even}(s)). \tag{58}$$

Alternatively, we have

$$\prod_{n=1}^{\infty}(1 - \lambda_{n,Sp}(s)) = \prod_{n=1}^{\infty}(1 - \lambda_{2n,U}(s)) \tag{59}$$

and

$$\prod_{n=1}^{\infty}(1 - \lambda_{n,SO,even}(s)) = \prod_{n=1}^{\infty}(1 - \lambda_{2n-1,U}(s)) \tag{60}$$

provided that the $\lambda_{n,U}(s)$ are indexed so that an even index n corresponds to an even eigenfunction and an odd index n is for an odd eigenfunction of the integral operator (49) with kernel $K_U(x, y) = S(x - y)$. These formulae can be used to give expressions for $\nu_G(j, s)$ in terms of the eigenvalues $\lambda_{n,U}(s)$.

9.6 The differential equation

We follow Mehta's treatment of how a Painlevé equation can be used to compute the nearest neighbor spacing.

Let $K(x,y) = \frac{\sin(x-y)}{x-y}$,

$$A(t) := \sum_{n=0}^{\infty} z^{n+1} \int_{[-t,t]^n} K(t,x_1)K(x_1,x_2)\dots K(x_{n-1},x_n)K(x_n,t)\; dx_1\dots dx_n$$

and

$$B(t) := \sum_{n=0}^{\infty} z^{n+1} \int_{[-t,t]^n} K(t,x_1)K(x_1,x_2)\dots K(x_{n-1},x_n)K(x_n,-t)\; dx_1\dots dx_n.$$

Then

$$A'(t) = 2B(t)^2. \tag{61}$$

This equation, originally discovered by Gaudin, is one equation of a non-linear system of differential equations satisfied by A and B. To see (61) we note that

$$A'(t) = A_1(t) + A_2(t) + A_3(t) + A_4(t)$$

where

$$A_1(t) := \sum_{n=1}^{\infty} z^{n+1} \int_{[-t,t]^n} \frac{\partial K(t,x_1)}{\partial t} K(x_1,x_2)\dots K(x_n,t)\; dx_1\dots dx_n$$

$$A_2(t) := \sum_{n=1}^{\infty} z^{n+1} \int_{[-t,t]^n} K(t,x_1)K(x_1,x_2)\dots K(x_{n-1},x_n)\frac{\partial K(x_n,t)}{\partial t}\; dx_1\dots dx_n$$

$$A_3(t) := \sum_{n=1}^{\infty} z^{n+1} \sum_{j=1}^{n} \int_{[-t,t]^{n-1}} K(t,x_1)\dots K(x_{j-2},x_{j-1})K(x_{j-1},t)K(t,x_{j+1}) \times K(x_{j+1},x_{j+2})\dots K(x_n,t)\; dx_1\dots dx_{j-1}dx_{j+1}\dots dx_n$$

$$A_4(t) := \sum_{n=1}^{\infty} z^{n+1} \sum_{j=1}^{n} \int_{[-t,t]^{n-1}} K(t,x_1)\dots K(x_{j-2},x_{j-1})K(x_{j-1},-t)K(-t,x_{j+1})$$
$$\times K(x_{j+1},x_{j+2})\dots K(x_n,t)\ dx_1\dots dx_{j-1}dx_{j+1}\dots dx_n$$

Notice that

$$\frac{\partial K(t,x)}{\partial t} = -\frac{\partial K(t,x)}{\partial x}.$$

In A_2 we replace $\frac{\partial K(t-x_n)}{\partial t}$ by $-\frac{\partial K(t-x_n)}{\partial x_n}$ and then integrate by parts with respect to x_n:

$$\begin{aligned}
A_2(t) : &= \sum_{n=1}^{\infty} z^{n+1} \int_{[-t,t]^n} K(t,x_1)K(x_1,x_2)\dots K(x_{n-1},x_n)\frac{\partial K(x_n,t)}{\partial t}\ dx_1\dots dx_n \\
&= -\sum_{n=1}^{\infty} z^{n+1} \int_{[-t,t]^n} K(t,x_1)K(x_1,x_2)\dots K(x_{n-1},x_n)\frac{\partial K(x_n,t)}{\partial x_n}\ dx_1\dots dx_n \\
&= -\sum_{n=1}^{\infty} z^{n+1} \int_{[-t,t]^{n-1}} K(t,x_1)K(x_1,x_2)\dots K(x_{n-1},x_n)K(x_n,t)\bigg|_{x_n=-t}^{t}\ dx_1\dots dx_{n-1} \\
&\qquad + \sum_{n=1}^{\infty} z^{n+1} \int_{[-t,t]^n} K(t,x_1)K(x_1,x_2)\dots \frac{\partial K(x_{n-1},x_n)}{\partial x_n} K(x_n,t)\ dx_1\dots dx_n.
\end{aligned}$$

We now repeat this integrating-by-parts process and, using the fact that $K(x,x)=1$, we obtain

$$\begin{aligned}
A_2(t) &= -\sum_{n=1}^{\infty} z^{n+1} \sum_{j=1}^{n} \int_{[-t,t]^{n-1}} K(t,x_1)K(x_1,x_2)\dots K(x_n,t)\bigg|_{x_j=-t}^{t}\ dx_1\dots dx_{j-1}dx_{j+1}dx_n \\
&\qquad + \sum_{n=1}^{\infty} z^{n+1} \int_{[-t,t]^n} \frac{\partial K(t,x_1)}{\partial x_1} K(x_1,x_2)\dots K(x_n,t)\ dx_1\dots dx_n \\
&= -A_1(t) - A_3(t) + A_4(t).
\end{aligned}$$

Thus,

$$A'(t) = 2A_4(t).$$

But now, using the symmetry $K(x,x') = K(x',x)$ and substituting $y_1 = x_n$,

$y_2 = x_{n-1}$ etc., we have

$$\begin{aligned} A_4(t) &= \sum_{n=1}^{\infty} z^{n+1} \sum_{j=1}^{n} \int_{[-t,t]^{n-1}} K(t,x_1)\ldots K(x_{j-2},x_{j-1})K(x_{j-1},-t)K(-t,x_{j+1}) \\ &\qquad\qquad \times K(x_{j+1},x_{j+2})\ldots K(x_n,t)\; dx_1\ldots dx_{j-1}dx_{j+1}\ldots dx_n \\ &= \sum_{j=1}^{\infty} z^j \int_{[-t,t]^{j-1}} K(t,x_1),\ldots K(x_{j-1},-t)\; dx_1\ldots dx_{j-1} \\ &\qquad \times \sum_{n=j}^{\infty} z^{n-j+1} \int_{[-t,t]^{n-j}} K(t,y_1)\ldots K(y_{n-j},-t)dy_1\ldots dy_{n-j}. \end{aligned}$$

If we let $m = n - j$ and then change j into $j+1$ we see that $A_4(t) = B(t)^2$ and so (61) is verified. The above argument used just a few properties of K, namely that it is symmetric and equal to 1 on the diagonal.

Now we let

$$S_{\pm}(t) = \sum_{n=0}^{\infty} z^n \int_{[-1,1]^n} k(1,x_1)k(x_1,x_2)\ldots k(x_{n-1},x_n)e^{\pm itx_n}\; dx_1\ldots dx_n.$$

where

$$k(x,x') = \frac{\sin t(x-x')}{x-x'}.$$

We claim that

$$zS_+S_- = (tA)' \qquad z(S_+^2 + S_-^2) = (tB)' \qquad \frac{z}{4i}(S_+^2 - S_-^2) = tB \tag{62}$$

where the differentiations are with respect to t. Note that

$$tA = \sum_{n=0}^{\infty} z^{n+1} \int_{[-1,1]^n} k(1,x_1)k(x_1,x_2)\ldots k(x_{n-1},x_n)k(x_n,1)\; dx_1\ldots dx_n.$$

Similarly

$$tB = \sum_{n=0}^{\infty} z^{n+1} \int_{[-1,1]^n} k(1,x_1)k(x_1,x_2)\ldots k(x_{n-1},x_n)k(x_n,-1)\; dx_1\ldots dx_n.$$

Now

$$(tA)' = \sum_{n=0}^{\infty} z^{n+1} \sum_{j=1}^{n+1} \int_{[-1,1]^n} k(1,x_1)\ldots \frac{\partial k(x_{j-1},x_j)}{\partial t}\ldots k(x_n,1)\; dx_1\ldots dx_n$$

Now $\frac{\partial k(x_{j-1},x_j)}{\partial t} = \cos(x_{j-1}-x_j)t = \frac{1}{2}(e^{i(x_{j-1}-x_j)t}+e^{-i(x_{j-1}-x_j)t})$; then we have 2 terms to consider in calculating $(tA)'$. One such term is

$$\frac{z}{2}\sum_{n=0}^{\infty}\sum_{j=1}^{n+1} z^{j-1}\int_{[-t,t]^{j-1}} k(1,x_1)\dots k(x_{j-1},x_j)e^{-ix_{j-1}t}\ dx_1\dots dx_{j-1}$$
$$\times z^{n+1-j}\int_{[-t,t]^{n+1-j}} k(1,x_n)\dots k(x_n,x_{n-1})\dots k(x_{j+1},x_j)e^{ix_jt}\ dx_n\dots dx_j$$
$$= \frac{z}{2}S_+(t)S_-(t).$$

The other term gives the same and we have $(tA)' = zS_+S_-$. Next

$$(tB)' = \sum_{n=0}^{\infty} z^{n+1}\sum_{j=1}^{n+1}\int_{[-1,1]^n} k(1,x_1)\dots\frac{\partial k(x_{j-1},x_j)}{\partial t}\dots k(x_n,-1)\ dx_1\dots dx_n$$

Observe that

$$\sum_{n=0}^{\infty} z^n\int_{[-1,1]^n} k(-1,x_1)k(x_1,x_2)\dots k(x_{n-1},x_n)e^{\pm itx_n}\ dx_1\dots dx_n$$
$$= \sum_{n=0}^{\infty} z^n\int_{[-1,1]^n} k(1,x_1)k(x_1,x_2)\dots k(x_{n-1},x_n)e^{\mp itx_n}\ dx_1\dots dx_n$$
$$= S_{\mp}(t).$$

Thus, we get $(tB)' = z(S_+^2+S_-^2)$. Finally we'll show that $(S_+^2-S_-^2) = tB$. Consider

$$S_+^2 = \sum_{\ell,m\geq 0} z^{\ell+m}\int_{[-1,1]^{\ell+m}} k(1,x_1)\dots k(x_{\ell-1},x_\ell)e^{itx_\ell}k(1,y_1)\dots k(y_{m-1},y_m)e^{ity_m}dx_1\dots dy_m.$$

If we change all the y_j into $-y_j$ and use the evenness of k we can rewrite this as

$$S_+^2 = \sum_{\ell,m\geq 0} z^{\ell+m}\int_{[-1,1]^{\ell+m}} k(1,x_1)\dots k(x_{\ell-1},x_\ell)e^{itx_\ell}k(-1,y_1)\dots k(y_{m-1},y_m)e^{-ity_m}dx_1\dots dy_m.$$

Thinking of t as real and taking the imaginary part leads to

$$S_+^2 - S_-^2 = \sum_{\ell,m\geq 0} z^{\ell+m}\int_{[-1,1]^{\ell+m}} k(1,x_1)\dots k(x_{\ell-1},x_\ell)$$
$$\times 2i(x_\ell - y_m)k(x_\ell,y_m)k(-1,y_1)\dots k(y_{m-1},y_m)dx_1\dots dy_m.$$

Renaming the variables leads to

$$S_+^2 - S_-^2 = \sum_{n\geq 0} z^n \int_{[-1,1]^n} k(1,x_1)k(x_1,x_2)\ldots k(x_{n-1},x_n)k(x_n,-1) \times \sum_{j=1}^{n+1} 2i(x_{j-1}-x_j)dx_1\ldots dx_n$$

where, in the inner sum we think of $x_0 = 1$ and $x_{n+1} = -1$. The inner sum telescopes to $4i$ and so we have

$$tB = \frac{z}{4i}(S_+^2 - S_-^2).$$

This is the last equation of (62).

Altogether we now have $A = 2B^2$; $zS_+S_- = (tA)'$; $z(S_+^2 + S_-^2) = (tB)'$; and $\frac{z}{4i}(S_+^2 - S_-^2) = tB$. We can use these to get a single differential equation for A. Using these equations together with $(S_+ + S_-)^2 - (S_+ - S_-)^2 = 4S_+S_-$ we have

$$(tA' + A)^2 = (tB' + B)^2 + 4(tB)^2.$$

Differentiating $A' = 2B^2$ leads to

$$tA'' + 2A' = 4B(tB' + B).$$

Now eliminating B from the last two equations gives

$$(tA'' + 2A')^2 + 4t^2(A')^2 - 8A'(tA' + A)^2 = 0 \qquad (63)$$

This is a Painlevé V equation.

From the definitions of A and B and from (61) we have the initial conditions

$$A(0) = z \qquad \text{and} \qquad A'(0) = 2z^2.$$

From these and (63) we can get a recurrence relation for the coefficients in the power series expansion about $t = 0$ for $A(t)$. With

$$A(t) = \sum_{n=0}^{\infty} a_n(z)t^n$$

we have

$$a_0 = z, \qquad a_1 = 2z^2,$$

and for $n \geq 2$,

$$\begin{aligned}
a_n = (-8a_0^2 n + 4a_1(n-1)n + 8a_1 n)^{-1} \Bigg(& 16 \sum_{j=0}^{n-2} \sum_{k=0}^{n-j-2} (j+1)(k+1) a_{j+1} a_{k+1} a_{n-j-k-2} \\
& +8 \sum_{j=0}^{n-3} \sum_{k=0}^{n-j-3} (j+1)(k+1) a_{j+1} a_{k+1} (n-j-k-2) a_{n-j-k-2} \\
& +8 \sum_{j=0}^{n-2} \sum_{k=0}^{n-j-1} (j+1) a_{j+1} a_k a_{n-j-k-1} - 4 \sum_{j=0}^{n-3} (j+1) a_{j+1} (n-j-2) a_{n-j-2} \\
& - \sum_{j=0}^{n-3} (j+1)(j+2) a_{j+2} (n-j-2)(n-j-1) a_{n-j-1} \\
& -4 \sum_{j=1}^{n-2} (j+1) a_{j+1} (n-j) a_{n-j} - 4 \sum_{j=1}^{n-2} (j+1) a_{j+1} (n-j-1)(n-j) a_{n-j} \Bigg)
\end{aligned}$$

10 Painlevé equations

10.1 Introduction

Painlevé equations are second order non-linear differential equations that play a useful role in describing certain quantities that arise in Random Matrix Theory, especially those quantities that are described by Fredholm determinants. For example, the scaled limit of the nearest neighbor distribution for $U(N)$, and the lowest eigenvalue of $SO(2N)$ and $USp(2N)$. In addition, the leading order term of the $2k$th moment $|\Lambda_X'(1)|^{2k}$ averaged over $U(N)$ is best expressed as the solution of a Painlev'e equation.

There are 6 basic types of Painlevè equations; for example PII is the equation

$$q'' = 2q^3 + tq + \alpha$$

for $q = q(t)$. An interesting fact is that if $A(t)$ satisfies the differential equation

$$A''(t) = -\frac{1}{2}tA(t)$$

then $q_0(t) = \frac{A'(t)}{A(t)}$ satisfies the PII equation with $\alpha = -1/2$. To see this we calculate

$$\begin{aligned}\left(\frac{A'}{A}\right)'' - 2\left(\frac{A'}{A}\right)^3 - t\left(\frac{A'}{A}\right) &= \frac{2(A')^3 - 3AA'A'' + A^2A'''}{A^3} - 2\frac{(A')^3}{A^3} - \frac{tA'}{A} \\ &= \frac{-3A'A'' + AA''' - tAA'}{A^2}\end{aligned}$$

Using $A'' = -tA/2$ and $A''' = -tA'/2 - A/2$ we see that

$$-3A'A'' + AA''' - tAA' = 3tAA'/2 - tAA'/2 - A^2/2 - tAA' = -A^2/2.$$

Interestingly, $\sigma = A'/A$ satisfies another second order non-linear equation:

$$(\sigma'')^2 + 4(\sigma')^3 + 2t(\sigma')^2 - 2\sigma\sigma' = \frac{1}{4}.$$

Let us just verify this assertion. Inserting $\sigma' = \frac{AA''-(A')^2}{A^2}$ and $\sigma'' = \frac{2(A')^3-3AA'A''+A^2A'''}{A^3}$ into the above and clearing denominators, we have to verify that

$$\begin{aligned}\frac{A^4}{4} &= 2A(A')^3 + 2t(A')^4 - 2A^2A'A'' - 4tA(A')^2A'' + 2tA^2(A'')^2 - 3(A')^2(A'')^2 \\ &\quad +4A(A'')^3 + 4(A')^3A''' - 6AA'A''A''' + A^2(A''')^2\end{aligned}$$

Differentiating the equation $A'' = -tA/2$ we have

$$A''' = -tA'/2 - A/2.$$

Substituting these expressions for A'' and A''' into the above, we find that the complicated equation above does indeed hold!

The structure of the Painlev/‘e equations is such that we can use one solution to generate an infinite sequence of solutions to the same equation but with changing parameters. With our function A above satisfying $A'' = -tA/2$, consider the ‘double Wronskian’ determinant

$$\tau_N(t) = \det_{N\times N}(A^{(i+j)}(t))|_{0\le i,j\le N-1}$$

formed with the function A satisfying the differential equation $A'' = -tA/2$ as above. Then $\sigma_N(t) = \tau_N'(t)/\tau_N(t)$ satisfies the differential equation

$$(\sigma_N'')^2 + 4(\sigma_N')^3 + 2t(\sigma_N')^2 - 2\sigma_N\sigma_N' = \frac{N^2}{4}$$

for $N = 1, 2, 3 \ldots$..

The verification we just did above is the case $N = 1$ of this fact. The goal of this chapter is to prove the general assertion for all N. To do this, we will make use of a surprising recursion that τ_N satisfies, namely

$$\tau_{N-1}\tau_{N+1} = \tau_N''\tau_N - (\tau_N')^2.$$

We will define a sequence of functions H_N which also satisfy this recursion and which satisfy the σ_N differential equation. It will then follow that, up to a constant, σ_N satisfies this differential equation. Finally, a major task will be to identify this constant.

Let us examine for a moment the recursion formula for τ_N. We call this **Amazing Fact 1.** Suppose that $A(t)$ is infinitely differentiable. Define $\tau_0(t) = 1$ and for $N \geq 1$,

$$\tau_N(t) = \det_{N\times N}(A^{(i+j)}(t))|_{0\leq i,j\leq N-1}.$$

Then

$$\tau_{N-1}\tau_{N+1} = \tau_N''\tau_N - (\tau_N')^2.$$

The differentiation is with respect to t. The first few cases are

$$\begin{aligned}
\tau_0(t) &= 1 \\
\tau_1(t) &= A(t) \\
\tau_2(t) &= \begin{vmatrix} A(t) & A'(t) \\ A'(t) & A''(t) \end{vmatrix} = A(u)A''(u) - A'(u)^2 \\
\tau_3(t) &= \begin{vmatrix} A & A' & A'' \\ A' & A'' & A''' \\ A'' & A''' & A^{(4)} \end{vmatrix} = AA''A^{(4)} - (A')^2A^{(4)} - A(A''')^2 + 2A'A''A''' - (A'')^3
\end{aligned}$$

which can easily be verified to satisfy the above recurrence. We will prove this fact shortly.

Thus, the complicated determinant $\tau_N(t)$ can be computed in a simpler way using this recursion formula. We will later see that computing a series solution for $\sigma_N = \tau_N'/\tau_N$ from the differential equation is even faster.

10.2 A recursion formula for determinants

We now prove the remarkable recursion formula for the determinants with derivatives, sometimes known as 'double Wronskians'.

There are two main ingredients. The first is a formula for determinants found by Lewis Carroll. It is

$$\begin{vmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \dots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{vmatrix} \times \begin{vmatrix} a_{1,1} & \dots & a_{1,n-2} \\ \vdots & \dots & \vdots \\ a_{n-2,1} & \dots & a_{n-2,n-2} \end{vmatrix}$$

$$= \begin{vmatrix} a_{1,1} & \dots & a_{1,n-1} \\ \vdots & \dots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,n-1} \end{vmatrix} \times \begin{vmatrix} a_{1,1} & \dots & a_{1,n-2} & a_{1,n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n-2,1} & \dots & a_{n-2,n-2} & a_{n-2,n} \\ a_{n,1} & \dots & a_{n,n-2} & a_{n,n} \end{vmatrix}$$

$$- \begin{vmatrix} a_{1,1} & \dots & a_{1,n-2} & a_{1,n-1} \\ \vdots & \vdots & \dots & \vdots \\ a_{n-2,1} & \dots & a_{n-2,n-2} & a_{n-2,n-1} \\ a_{n,1} & \dots & a_{n,n-2} & a_{n,n-1} \end{vmatrix} \times \begin{vmatrix} a_{1,1} & \dots & a_{1,n-2} & a_{1,n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n-2,1} & \dots & a_{n-2,n-2} & a_{n-2,n} \\ a_{n-1,1} & \dots & a_{n-1,n-2} & a_{n-1,n} \end{vmatrix}$$

A shorthand for this formula is (notation from Kujiwara et al)

$$D \times D \begin{pmatrix} n-1 & n \\ n-1 & n \end{pmatrix} = D \begin{pmatrix} n-1 \\ n-1 \end{pmatrix} \times D \begin{pmatrix} n \\ n \end{pmatrix} - D \begin{pmatrix} n-1 \\ n \end{pmatrix} \times D \begin{pmatrix} n \\ n-1 \end{pmatrix}$$

where D is the full $n \times n$ determinant and $D\left(\begin{smallmatrix} a & b & c & \dots \\ x & y & z & \dots \end{smallmatrix}\right)$ means the determinant but with the rows $a, b, c, \dots$ and columns $x, y, z, \dots$ removed. This formula is a special case of Sylvester's Theorem, for which see Mehta, page 37.

We apply Lewis Carroll's formula in the situation where our determinant D is a "double" Wronskian. The Wronskian of a set of functions $\{u_1, \dots u_n\}$ is

$$W(u_1, \dots, u_n) = \begin{vmatrix} u_1 & u_2 & \dots & u_n \\ u_1' & u_2' & \dots & u_n' \\ \vdots & \vdots & \dots & \vdots \\ u_n^{(n-1)} & u_2^{(n-1)} & \dots & u_n^{(n-1)} \end{vmatrix}$$

Here the u_j are functions of t say and the differentiation is with respect to t. Wronskians appear in linear algebra and differential equations when one is trying to determine if a set of functions are linearly independent; they are if and only if the Wronskian is non-zero.

In our situation, the "double" Wronskian means that we start with one function $u_1 = A(t)$ and then have $u_2 = A'(t)$, $u_3 = A''(t)$, etc. In this way, we have

$$W(A, A', \ldots, A^{(n-1)}) = \tau_n(t)$$

using our earlier notation. If we now apply Lewis Carroll's identity to the determinant $W(A, A', \ldots, A^{(n-1)})$ we get

$$D = \tau_n(t) \qquad D\begin{pmatrix} n \\ n \end{pmatrix} = \tau_{n-1}(t) \qquad D\begin{pmatrix} n-1 \ n \\ n-1 \ n \end{pmatrix} = \tau_{n-2}(t).$$

Next, we observe that if we have an $n \times n$ determinant of functions of some variable, say t, and we differentiate it, the the result is a sum of n determinants each which looks like the original one except that each column is differentiated in turn. Thus,

$$\frac{d}{dt}\begin{vmatrix} u_{11} & u_{12} & \ldots & u_{1n} \\ u_{21} & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ u_{n1} & u_{n2} & \ldots & u_{nn} \end{vmatrix} = \begin{vmatrix} u'_{11} & u_{12} & \ldots & u_{1n} \\ u'_{21} & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ u'_{n1} & u_{n2} & \ldots & u_{nn} \end{vmatrix}$$
$$+ \begin{vmatrix} u_{11} & u'_{12} & \ldots & u_{1n} \\ u_{21} & u'_{22} & \ldots & u_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ u_{n1} & u'_{n2} & \ldots & u_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} u_{11} & u_{12} & \ldots & u'_{1n} \\ u_{21} & u_{22} & \ldots & u'_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ u_{n1} & u_{n2} & \ldots & u'_{nn} \end{vmatrix}$$

If we apply this to the Wronskian of $A, A', \ldots,$ then most of the determinants above are zero because they have two identical columns. Thus, we have

$$W'(A, A', \ldots, A^{(n-1)}) = W(A, A', \ldots, A^{(n-2)}, A^{(n)}) = D\begin{pmatrix} n \\ n-1 \end{pmatrix} = D\begin{pmatrix} n-1 \\ n \end{pmatrix}.$$

the last equality follows because the determinant of a matrix and its transpose are equal. Thus,

$$D\begin{pmatrix} n \\ n-1 \end{pmatrix} = D\begin{pmatrix} n-1 \\ n \end{pmatrix} = \tau'_n(t).$$

Finally, if we differentiate

$$\tau_n'(t) = \frac{d}{dt} D\begin{pmatrix} n-1 \\ n \end{pmatrix} = \begin{vmatrix} A & \dots & A^{(n-3)} & A^{(n-2)} \\ \vdots & \vdots & \dots & \vdots \\ A^{(n-3)} & \dots & A^{(2n-6)} & A^{(2n-5)} \\ A^{(n-1)} & \dots & A^{(2n-3)} & A^{(2n-2)} \end{vmatrix}$$

we see that only the last column gets differentiated, so that

$$\tau_n'(t) = \begin{vmatrix} A & \dots & A^{(n-3)} & A^{(n-1)} \\ \vdots & \vdots & \dots & \vdots \\ A^{(n-3)} & \dots & A^{(2n-6)} & A^{(2n-4)} \\ A^{(n-1)} & \dots & A^{(2n-3)} & A^{(2n-1)} \end{vmatrix} = D\begin{pmatrix} n-1 \\ n-1 \end{pmatrix}.$$

Inserting these results into Lewis Carroll's formula, we obtain

$$\tau_n \tau_n'' - (\tau_n')^2 = \tau_{n-1}\tau_{n+1}.$$

It just remains to prove Lewis Carroll's formula.

10.3 Proof of Lewis Carroll's identity

We note that the identity can be put in a determinant form:

$$D \times D\begin{pmatrix} n-1 \; n \\ n-1 \; n \end{pmatrix} = \begin{vmatrix} D\binom{n-1}{n-1} & D\binom{n-1}{n} \\ D\binom{n}{n-1} & D\binom{n}{n} \end{vmatrix}$$

This form suggests Sylvester's identity which involves three integer parameters $h \leq m \leq n$; in Carroll's identity $h = n-1$ and $m = n-1$.

We prove the identity by induction. It is trivial for $n = 2$ and for $n = 3$ amounts to checking that

$$a \times \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} a & b \\ d & e \end{vmatrix} & \begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ \\ \begin{vmatrix} a & b \\ d & e \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{vmatrix}.$$

Now the thing to do when $n > 3$ is to use elementary row operations which do not change the value of any of the determinants but which change the first column to

$$\begin{matrix} a \\ 0 \\ \vdots \\ 0 \end{matrix}$$

Then it is required to verify that

$$\begin{vmatrix} a & \dots & b_{1,n} \\ \vdots & \dots & \vdots \\ 0 & \dots & b_{n,n} \end{vmatrix} \times \begin{vmatrix} a & \dots & b_{1,n-2} \\ \vdots & \dots & \vdots \\ 0 & \dots & b_{n-2,n-2} \end{vmatrix}$$

$$= \begin{vmatrix} a & \dots & b_{1,n-1} \\ \vdots & \dots & \vdots \\ 0 & \dots & b_{n-1,n-1} \end{vmatrix} \times \begin{vmatrix} a & \dots & b_{1,n-2} & b_{1,n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & \dots & b_{n-2,n-2} & b_{n-2,n} \\ 0 & \dots & b_{n,n-2} & b_{n,n} \end{vmatrix}$$

$$- \begin{vmatrix} a & \dots & b_{1,n-2} & b_{1,n-1} \\ \vdots & \vdots & \dots & \vdots \\ 0 & \dots & b_{n-2,n-2} & b_{n-2,n-1} \\ 0 & \dots & b_{n,n-2} & b_{n,n-1} \end{vmatrix} \times \begin{vmatrix} a & \dots & b_{1,n-2} & b_{1,n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & \dots & b_{n-2,n-2} & b_{n-2,n} \\ 0 & \dots & b_{n-1,n-2} & b_{n-1,n} \end{vmatrix}$$

But each determinant to be evaluated now is a times a determinant one dimension smaller than previously. When we divide out a^2 from this equation we are just left with an instance of the $n-1$ case of Lewis Carroll's identity. Thus, the identity follows by induction.

10.4 The recursive system of p's, q's, and H's

We begin with a function $q = q_0$ that satisfies

$$q'' = 2q^3 + tq + \alpha. \tag{64}$$

We introduce an auxiliary function p by

$$p = q' + q^2 + \frac{t}{2} \tag{65}$$

and then define H by

$$H = -\frac{1}{2}p(2q^2 - p + t) - (\alpha + \frac{1}{2})q. \tag{66}$$

Notice that

$$H'(t) = -\frac{1}{2}p(t).$$

Also, it is easy (especially with a computer algebra system) to verify that H satisfies

$$(H'')^2 + 4(H')^3 + 2t(H')^2 - 2HH' = \frac{(\alpha + 1/2)^2}{4}; \tag{67}$$

one just uses equations (64), (65), and (66) and two derivatives of (64) to plug into the left side of (67); after simplification it works out.

There is a lot of structure to this set up. For example, if we define

$$q_1 = -q + \frac{\frac{1}{1} - \alpha}{2q^2 + t - p},$$

then q_1 satisfies the same Painlevé equation as q does, but with α replaced by $\alpha - 1$, i.e.

$$q_1'' = 2(q_1)^3 + tq_1 + (\alpha - 1).$$

Then we can define

$$p_1 = q_1' + (q_1)^2 + \frac{t}{2}$$

and

$$H_1 = -\frac{1}{2}p_1(2q_1^2 - p_1 + t) - (\alpha - \frac{1}{2})q_1$$

and it follows from the earlier verification that H_1 satisfies the differential equation

$$(H_1'')^2 + 4(H_1')^3 + 2t(H_1')^2 - 2H_1H_1' = \frac{(\alpha - 1/2)^2}{4}.$$

Moreover, amazingly(!), we find that

$$H_1 - H = q,$$

$$q_1 - q = \frac{p_1'}{p_1},$$

and

$$H_1'(t) = -\frac{1}{2}p_1(t).$$

In the case mentioned at the start of this chapter, we have

$$q_0(t) = \frac{A'(t)}{A(t)}$$

which we shorten to $q_0 = A'/A$. This satisfies PII with $\alpha = -1/2$. Then we calculate $p_0 = 0$ and $H_0 = 0$, using only the fact that $A'' = -tA/2$. Proceeding further, we find that

$$q_1 = \frac{A^3 - tA^2A' - 2(A')^3}{tA^3 - 2A(A')^2}$$

which satisfies PII with $\alpha = -3/2$. Further,

$$p_1 = \frac{tA^2 + 2(A')^2}{A^2};$$

and

$$H_1 = \frac{A'}{A}.$$

We want to iterate this process. Let $\alpha_1 = \alpha$, $\alpha_2 = \alpha_1 - 1$ and, in general, $\alpha_n = \alpha_{n-1} - 1$ so that $\alpha_n = \alpha - (n-1)$. Given a function q_n that satisfies

$$q_n'' = 2q_n^3 + tq_n + \alpha_n \tag{68}$$

we define p_n and H_n by

$$p_n = q_n' + q_n^2 + \frac{t}{2} \tag{69}$$

and

$$H_n = -\frac{1}{2}p_n(2q_n^2 - p_n + t) - (\alpha_n + \frac{1}{2})q_n \tag{70}$$

and then it follows as above that

$$(H_n'')^2 + 4(H_n')^3 + 2t(H_n')^2 - 2H_nH_n' \quad = \quad \frac{(\alpha_n + 1/2)^2}{4}. \tag{71}$$

We can then proceed to define

$$q_{n+1} = -q_n + \frac{\frac{1}{2} - \alpha_n}{2q_n^2 + t - p_n},$$

and then q_{n+1} satisfies the Painlevé equation with α_{n+1}, i.e.

$$q_{n+1}'' = 2(q_{n+1})^3 + tq_{n+1} + \alpha_{n+1}.$$

Then let

$$p_{n+1} = q_{n+1}' + q_{n+1}^2 + \frac{t}{2} \tag{72}$$

and

$$H_{n+1} = -\frac{1}{2}p_{n+1}(2q_{n+1}^2 - p_{n+1} + t) - (\alpha_{n+1} + \frac{1}{2})q_{n+1} \tag{73}$$

and our three amazing equations hold:

$$H_{n+1} - H_n = q_n, \tag{74}$$

$$q_{n+1} - q_n = \frac{p_{n+1}'}{p_{n+1}}, \tag{75}$$

and

$$H_n'(t) = -\frac{1}{2}p_n(t).$$

10.5 The recursion for H

We eliminate p_N and q_N from our equations as follows. We have

$$H_{n+1} - 2H_n + H_{n-1} = q_n - q_{n-1} = \frac{p_n'}{p_n} = \frac{H_n''(t)}{H_n'(t)} = \frac{d}{dt}\log(\frac{d}{dt}H_n). \tag{76}$$

Now let

$$T_n(t) = \exp\left(\int_0^t H_n(u)\ du\right)$$

so that

$$H_n(t) = \frac{d}{dt}\log T_n(t) = \frac{T_n'(t)}{T_n(t)}.$$

Then (76) becomes

$$\frac{d}{dt}\log\frac{T_{n-1}(t)T_{n+1}(t)}{T_n(t)^2} = \frac{d}{dt}\log(\frac{d}{dt}H_n(t)) = \frac{d}{dt}\log\left(\frac{d^2}{dt^2}\log T_n(t)\right).$$

If we integrate both sides and exponentiate, we find that

$$\frac{T_{n-1}(t)T_{n+1}(t)}{T_n(t)^2} = C^*(n)\frac{T_n''(t)T_n(t) - T_n'(t)^2}{T_n(t)^2}$$

for some constant $C^*(n)$. Suppose that we have started from a situation with $\tau_0(t) = C(0)T_0(t)$ and $\tau_1(t) = C(1)T_1(t)$ and have the recursion formulas

$$T_{n+1}(t) = C^*(n)\frac{T_n''(t)T_n(t) - T_n'(t)^2}{T_{n-1}(t)}.$$

Then, it follows by induction that

$$\begin{aligned}\tau_{n+1}(t) &= \frac{\tau_n''(t)\tau_n(t) - \tau_n'(t)^2}{\tau_{n-1}(t)} = \frac{C(n)^2(T_n''(t)T_n(t) - T_n'(t)^2)}{C(n-1)T_{n-1}(t)}\\ &= \frac{C(n)^2C^*(n)}{C(n-1)}T_{n+1}(t) = C(n+1)T_{n+1}(t).\end{aligned}$$

Therefore,

$$\sigma_n(t) = \frac{\tau_n'(t)}{(n)\tau_n(t)} = \frac{C(n)T_n'(t)}{C(n)T_n(t)} = H_n(t)$$

and so $\sigma_n(t)$ satisfies the differential equation.

In order to finish this proof, we need to check that thee are constants $C(0)$ and $C(1)$ such that $\tau_0(t) = C(0)T_0(t)$ and $\tau_1(t) = C(1)T_1(t)$. By definition, $\tau_0(t) = 1$ and $\tau_1(t) = A(t)$. In order to calculate T_0 and T_1 we need to recall H_0 and H_1. We started from $q_0 = A'/A$ which satisfied the PII with $\alpha = -1/2$, and then defined in turn

$$p_0 = q_0' + q_0^2 + t/2 = 0$$

and

$$H_0 = -(1/2)p_0(2q_0^2 - p_0 + t) - (\alpha + 1/2)q_0 = 0.$$

Thus,

$$T_0(t) = \exp\left(\int_0^t H_0(u)\ du\right) = \exp(0) = 1 = \tau_0(t).$$

Further, as we already calculated in the first section, $H_1(t) = q_0(t) = A'(t)/A(t)$, so that

$$\begin{aligned} T_1(t) &= \exp\left(\int_0^t H_1(u)\ du\right) = \exp\left(\int_0^t \frac{A'(u)}{A(u)}\ du\right) = \exp(\log A(t) - \log A(0)) \\ &= \frac{A(t)}{A(0)} = C(1)\tau_1(t) \end{aligned}$$

with $C(1) = 1/A(0)$, which is fine as long as $A(0) \neq 0$.

Now we know that $\sigma_n(t)$ satisfies the differential equation (71), and we can use this to quickly find a series expansion for $\sigma_n(t)$. But we need to know what initial conditions to use. These will depend on the original function $A(t)$. Thus, we have

$$\tau_N(t) = \tau_N(0)\exp\left(\int_0^t \sigma_N(u)\ du\right)$$

and $\tau_N(0)$ and $\sigma_N'(0)$ will depend on $A(0)$ and $A'(0)$ in a way that we still have to determine.

10.6 Expressing $A^{(n)}(t)$ in terms of $A(t)$ and $A'(t)$

Recall that the differential equation for $A(t)$ is

$$A''(t) = -\frac{t}{2}A(t).$$

We can repeatedly differentiate this equation to find expressions for $A^{(n)}(t)$:

$$\begin{aligned}
A''' &= -\frac{1}{2}A - \frac{t}{2}A' \\
A^{(4)} &= -\frac{1}{2}A' - \frac{1}{2}A' - \frac{t}{2}A'' = -A' - \frac{t}{2}(-\frac{t}{2}A) = \frac{t^2}{4}A - A' \\
A^{(5)} &= -A'' + \frac{t}{2}A + \frac{t^2}{4}A' = tA + \frac{t^2}{4}A' \\
&\dots
\end{aligned}$$

Let us write

$$A^{(n)}(t) = f_n(t)A(t) + g_n(t)A'(t).$$

Then

$$\begin{aligned}
A^{(n+1)} &= f_n'A + f_nA' + g_n'A' + g_nA'' = f_n'A + f_nA' + g_n'A' - g_ntA/2 \\
&= (f_n' - \frac{t}{2}g_n)A + (f_n + g_n')A'
\end{aligned}$$

so that we have the linked recurrence formulas

$$\left\{ \begin{array}{c} f_{n+1}(t) \\ g_{n+1}(t) \end{array} \right\} = \left\{ \begin{array}{c} f_n'(t) - \frac{t}{2}g_n(t) \\ f_n(t) + g_n'(t) \end{array} \right\}$$

with initial conditions $f_0(t) = 1$ and $g_0(t) = 0$.

10.7 Power series solution for A

Assume that

$$A(t) = \sum_{n=0}^{\infty} a_n t^n$$

is a solution of $A'' = -tA/2$. Then we have that $a_2 = 0$ and the recurrence formula

$$a_{n+3} = -\frac{a_n}{2(n+2)(n+3)}$$

holds for $n \geq 0$. This leads to the general solution

$$
\begin{aligned}
A(t) &= a_0\left(1-\frac{t^3}{12}+\frac{t^6}{720}-\frac{t^9}{103680}+\dots\right)+a_1\left(t-\frac{t^4}{24}+\frac{t^7}{2016}-\frac{t^{10}}{362880}+\dots\right)\\
&= a_0\sum_{n=0}^{\infty}(-1)^n\prod_{j=1}^{n}(3j-2)\frac{t^{3n}}{(3n)!2^n}+a_1\sum_{n=0}^{\infty}(-1)^n\prod_{j=1}^{n}(3j-1)\frac{t^{3n+1}}{(3n+1)!2^n}
\end{aligned}
$$

Recall that the differential equation satisfied by the Airy function is $y'' = -ty$.

10.8 Power series solution for σ_N

Let us illustrate these computations. First we'll find a series solution for $\sigma(t) = \sigma_N(t)$ which satisfies

$$(\sigma'')^2 + 4(\sigma')^3 + 2t(\sigma')^2 - 2\sigma\sigma' = N^2/4.$$

Let

$$\sigma(t) = \sum_{n=0}^{\infty} h_n t^n.$$

Then using $\sigma'(t) = \sum(n+1)h_{n+1}t^n$ and $\sigma''(t) = \sum(n+1)(n+2)h_{n+2}t^n$ and substituting into $(\sigma'')^2 + 4(\sigma')^3 + 2t(\sigma')^2 - 2\sigma\sigma' = \frac{N^2}{4}$ we have

$$
\begin{aligned}
\frac{N^2}{4} = &\sum_{m,n}(m+1)(m+2)(n+1)(n+2)h_{m+2}h_{n+2}t^{m+n}\\
&+4\sum_{\ell,m,n}(\ell+1)(m+1)(n+1)h_{\ell+1}h_{m+1}h_{n+1}t^{\ell+m+n}\\
&+2\sum_{m,n}(m+1)(n+1)h_{m+1}h_{n+1}t^{m+n+1} - 2\sum_{m,n}(n+1)h_m h_{n+1}t^{m+n}.
\end{aligned}
$$

This gives

$$\frac{N^2}{4} = 4h_2^2 + 4h_1^3 - 2h_0h_1$$

and, for $k \geq 1$,

$$
\begin{aligned}
0 = &\sum_{m+n=k} ((m+1)(m+2)(n+1)(n+2)h_{m+2}h_{n+2} - 2(n+1)h_m h_{n+1}) \\
&+2 \sum_{m+n=k-1} (m+1)(n+1)h_{m+1}h_{n+1} \\
&+4 \sum_{\ell+m+n=k} (\ell+1)(m+1)(n+1)h_{\ell+1}h_{m+1}h_{n+1}.
\end{aligned}
$$

In the sums above, ℓ, m, and n are all non-negative. This gives the recursion

$$
\begin{aligned}
&4(k+1)(k+2)h_2 h_{k+2} \\
&\quad = -\sum_{\substack{m+n=k \\ mn\neq 0}} ((m+1)(m+2)(n+1)(n+2)h_{m+2}h_{n+2} - 2(n+1)h_m h_{n+1}) \\
&\qquad -2 \sum_{m+n=k-1} (m+1)(n+1)h_{m+1}h_{n+1} \\
&\qquad -4 \sum_{\ell+m+n=k} (\ell+1)(m+1)(n+1)h_{\ell+1}h_{m+1}h_{n+1}
\end{aligned}
$$

which allows one to solve for h_{k+2} knowing h_m for $1 \leq m \leq k+1$. From there, it is an easy matter to integrate the series for $\sigma(t)$ and then to exponentiate to find the series for $\tau_n(t)$.

Incidentally, the A that satisfies the differential equation above is $A(u) = \mathrm{Ai}(-2^{-1/3}u)$ where Ai is the Airy function.

10.9 Initial Conditions

Let's consider the initial conditions $A(0) = 0$ and $A'(0) = 1$ for our differential equation $A''(t) = -tA(t)/2$ and find the initial conditions for $\sigma(t)$. Basically want we want to know is the behavior near 0 of

$$\tau_N(t) = \det(A^{i+j}(t))|_{0\leq i,j\leq N-1}.$$

It is not difficult to see that this will have the shape

$$\tau_N(t) = \tau_N(0) +$$

11 Moments of Characteristic Polynomials

We define the characteristic polynomial of a matrix X by

$$\Lambda_X(s) = \prod_{n=1}^{N}(1 - se^{-i\theta_n}) \tag{77}$$

where X is a matrix with eigenvalues $e^{i\theta_n}$ for $n = 1, 2, \ldots, N$. This definition differs slightly from the usual definition of characteristic polynomial, but it does have the basic properties that it is a polynomial of degree N with zeros at the eigenvalues of the matrix X; but it is not necessarily monic.

It satisfies the functional equation

$$\Lambda_X(s) = \ldots \Lambda_{X^*}(1/s). \tag{78}$$

We are interested in evaluating, after Keating and Snaith, the moments

$$M_U(N, z) := \int_{U(N)} |\Lambda_X(1)|^z dX, \tag{79}$$

$$M_O(N, z) := \int_{O(2N)} \Lambda_X(1)^z dX, \tag{80}$$

and

$$M_S(N, z) := \int_{USp(2N)} \Lambda_X(1)^z dX, \tag{81}$$

where z is a complex variable.

At this point we will begin to use X for a variable matrix and dX for the appropriate Haar measure of the group in question, which should be clear from the context.

11.1 Selberg's Integral

In the 1930s Selberg found an amazing generalization of Euler's beta-integral. Let

$$\Delta(x) = \prod_{1 \le j < \ell \le n} (x_\ell - x_j)$$

Then

$$\int_0^1 \cdots \int_0^1 |\Delta(x)|^{2\gamma} \prod_{j=1}^{n} x_j^{\alpha-1}(1-x_j)^{\beta-1}\ dx_1 \ldots dx_n$$
$$= \prod_{j=0}^{n-1} \frac{\Gamma(1+\gamma+\gamma j)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+(n+j-1)\gamma)} \tag{82}$$

for $\Re\alpha > 0, \Re\beta > 0, \Re\gamma > -\min\{\frac{1}{n}, \frac{\Re\alpha}{n-1}, \frac{\Re\beta}{n-1}\}$. The case $n = 1$ is Euler's formula for the beta-function. For $n = 2$ it is

$$\int_0^1 \int_0^1 |x-y|^{2\gamma}(xy)^{\alpha-1}((1-x)(1-y))^{\beta-1}\ dx\ dy$$
$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \times \frac{\Gamma(1+2\gamma)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)}$$

A second form of Selberg's integral is

$$\int_{[-1,1]^n} |\Delta(x_1,\ldots,x_n)|^{2\gamma} \prod_{j=1}^{n} (1-x_j)^{\alpha-1}(1+x_j)^{\beta-1} dx_j$$
$$= 2^{\gamma n(n-1)+n(\alpha+\beta-1)} \prod_{j=0}^{n-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+\gamma(n+j-1))} \tag{83}$$

valid for $\Re\alpha > 0$, $\Re\beta > 0$, and $\Re\gamma > -\min\{\frac{1}{n}, \frac{\Re\alpha}{n-1}, \frac{\Re\beta}{n-1}\}$.

A third version of Selberg's integral formula is

$$\int_{R^n} |\Delta(x_1,\ldots,x_n)|^{2\gamma} \prod_{j=1}^{n} (a+ix_j)^{-\alpha}(b-ix_j)^{-\beta}\ dx_j$$
$$= \frac{(2\pi)^n}{(a+b)^{(\alpha+\beta)n-\gamma n(n-1)-n}} \prod_{j=0}^{n-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+\beta-(n+j-1)\gamma-1)}{\Gamma(1+\gamma)\Gamma(\alpha-j\gamma)\Gamma(\beta-j\gamma)} \tag{84}$$

valid for $\Re a > 0$, $\Re b > 0$, $\Re\alpha > 0$, $\Re\beta > 0$, $\Re(\alpha+\beta) > 1$, and

$$-\frac{1}{n} < \Re\gamma < \frac{1}{n-1} \min\{\Re\alpha, \Re\beta, \tfrac{1}{2}\Re(\alpha+\beta-1)\}.$$

11.2 Moments of characteristic polynomials

Keating and Snaith showed how to use Selberg's integral to find an exact formula for any moment of the absolute value of the characteristic polynomials averaged over $U(N)$.

Theorem (Keating and Snaith 1998) Let

$$g(K,N) := \int_{U(N)} |\Lambda_U(1)|^{2K}\ dU_N = \int_{U(N)} |\det(I-U)|^{2K}\ dU_N.$$

Then, for $\Re K > -1/2$,

$$g(K,N) \quad = \quad \prod_{j=1}^{N} \frac{\Gamma(j+2K)\Gamma(j)}{\Gamma(j+K)^2} \tag{85}$$

We now give the derivation of (85) from Selberg's integral. First observe that

$$|e^{i\alpha} - e^{i\beta}| = 2|\sin(\alpha/2 - \beta/2)|. \tag{86}$$

Thus,

$$\begin{aligned} g(K,N) &= \frac{2^{N(N-1)+2KN}}{(2\pi)^N N!} \int_{[0,2\pi]^N} \prod_{1\le j<k\le N} |\sin(\theta_k/2 - \theta_j/2)|^2 \prod_{n=1}^{N} |\sin(\theta_n/2)|^{2K}\ d\theta_n \\ &= \frac{2^{N^2+2KN}}{(2\pi)^N N!} \int_{[0,\pi]^N} \prod_{1\le j<k\le N} |\sin(\theta_k - \theta_j)|^2 \prod_{n=1}^{N} |\sin(\theta_n)|^{2K}\ d\theta_n \end{aligned}$$

Now using $\sin(\alpha-\beta) = (\cot\alpha - \cot\beta)\sin\alpha\sin\beta$, we find that

$$g(K,N) = \frac{2^{N^2+2KN}}{(2\pi)^N N!} \int_{[0,\pi]^N} \prod_{1\le j<k\le N} |\cot\theta_k - \cot\theta_j)|^2 \prod_{n=1}^{N} |\sin(\theta_n)|^{2K+2N-2}\ d\theta_n.$$

Now let $x_n = \cot\theta_n$. Then, by (84),

$$\begin{aligned} g(K,N) &= \frac{2^{N^2+2KN}}{(2\pi)^N N!} \int_{(-\infty,\infty)^N} |\Delta(x_1,\ldots,x_n)|^2 \prod_{n=1}^{N} |(1+ix_n)(1-ix_n)|^{-N-K}\ dx_n \\ &= \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(2K+j)}{\Gamma(j+K)^2}. \end{aligned}$$

11.3 More formulas for $g(K,N)$

It can be shown that

$$\begin{aligned} g(K,N) &= \frac{1}{N!(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{n=1}^{N}|1-e^{i\theta_n}|^{2K}\prod_{1\le m<n\le N}|e^{i\theta_n}-e^{i\theta_m}|^2\,d\theta_1\ldots\,d\theta_N \\ &= \prod_{i,j=1}^{K}\frac{N+i+j-1}{i+j-1} \\ &= \prod_{j=0}^{K-1}\frac{j!}{(j+K)!}(N+1)(N+2)^2\ldots(N+K)^K\ldots(N+2K-2)^2(N+2K-1) \\ &= \frac{(N+1)(N+2)^2\ldots(N+K)^K\ldots(N+2K-2)^2(N+2K-1)}{1\cdot 2^2\cdot 3^3\cdot\ldots\cdot K^K\cdot(K+1)^{K-1}\cdot\ldots\cdot(2K-1)} \\ &= \frac{G(N+2K+1)G(N+1)}{G(N+K+1)^2}\frac{G(K+1)^2}{G(2K+1)}. \end{aligned}$$

The first formula is just the definition and the next three formulas are simple rearrangements of each other. Note that in the definition N must be an integer but K is free to be any complex number, while in the first three formulas K must be an integer while N is free to be any complex number. In the last formula G is the Barnes double gamma function which is defined by

$$G(1+z) = (2\pi)^{z/2}e^{-(z+1)z/2-\gamma z^2/2}\prod_{k=1}^{\infty}(1+\frac{z}{k})^k e^{-z+\frac{z^2}{2k}}$$

and satisfies

$$G(z+1) = \Gamma(z)G(z).$$

Both K and N are free to be complex numbers; this formula follows immediately from (85) by appropriate use of (88) below.

Note that

$$g(K,N)\sim g_K\frac{N^{K^2}}{K^2!}$$

where

$$g_K = \frac{(K^2)!}{1\cdot 2^2\cdot\ldots\cdot K^K\cdot(K+1)^{K-1}\cdot\ldots\cdot(2K-1)}$$

We calculate

$$g_1 = 1 \quad g_2 = 2 \quad g_3 = 42 \quad g_4 = 24024$$

We record the definition and basic properties of G. It is defined by the Hadamard product

$$G(z+1) = (2\pi)^{\frac{z}{2}} \exp\left(-\tfrac{1}{2}(z^2 + \gamma z^2 + z)\right) \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^n e^{-z + \frac{z^2}{2n}} \tag{87}$$

where γ is Euler's constant. We immediately see that $G(z+1)$ is entire of order two and that it has a zero at $z = -n$ of order n. Further

$$G(z+1) = \Gamma(z) G(z) \tag{88}$$

hence the name double-gamma function. The logarithm of $G(z+1)$ has a nice Taylor expansion:

$$\log G(z+1) = \tfrac{1}{2}(\log 2\pi - 1)z - \tfrac{1}{2}(1+\gamma)z^2 + \sum_{n=3}^{\infty} (-1)^{n-1} \zeta(n-1) \frac{z^n}{n} \tag{89}$$

valid for $|z| < 1$.

For large $|z|$ with $|\arg(z)| < \pi$ we have the asymptotic formula

$$\log G(z+1) = z^2(\tfrac{1}{2}\log z - \tfrac{3}{4}) + \tfrac{1}{2} z \log z - \frac{1}{12}\log z + \zeta'(-1) + O(1/|z|). \tag{90}$$

Some other identities of a combinatorial nature include

$$\begin{aligned} g(K,N) &= \det \binom{2K}{K+j-i}_{1 \le i,j \le N} \\ &= \det \binom{2K-1+N+j-i}{2K-1}_{1 \le i,j \le N} \\ &= \prod_{j=1}^{K} \prod_{n=1}^{N} \frac{2K+n-j}{N+K+1-j-n} \end{aligned}$$

Also,

$$g(K,N) = \det_{K \times K} \binom{N+K+j-1}{K+i-1}.$$

In the course of developing below a formula for "shifted" moments, or autocorrelations, of characteristic polynomials we will see that

$$g(K,N) \quad = \quad S_{(N\ldots N)}(1,\ldots,1) \tag{91}$$

where S is the Schur polynomial with $2K$ ones and K N's, and then that

$$g(K,N) \quad = \quad \frac{(-1)^K}{(2\pi i)^{2K}K!^2}\oint\cdots\oint\frac{e^{\frac{N}{2}\sum_{j=1}^{K}(z_j-z_{K+j})}\Delta(z_1,\ldots,z_{2K})^2}{\prod_{1\le i,j\le K}(1-e^{-(z_i-z_{K+j})})\prod_{i=1}^{2K}z_i^{2K}}\,dz_1\ldots\,dz_{2K}$$

where the paths of integration are small circles around the origin.

The leading order asymptotics for large N and fixed K can be expressed as a K-fold integral:

$$g(K,N)\sim\frac{1}{K!(2\pi i)^K}\int_{|z_1|=1}\cdots\int_{|z_K|=1}\frac{e^{N\sum_{j=1}^{K}z_i}\Delta(z)^2}{\prod_{j=1}^{K}z_j^{2K}}dz_1\ldots dz_K. \tag{92}$$

11.4 The prime factorization of g_K

Conrey and Farmer proved that g_K is always an integer (but just barely so!). For example

$$\begin{aligned} g_{100} = {} & 2^{95}\cdot3^{65}\cdot5^{24}\cdot7^{33}\cdot11^{10}\cdot13^{33}\cdot17^{36}\cdot19^{29}\cdot23^{20}\cdot29^{16}\cdot31^{11}\cdot37^{10}\cdot41^{12}\\ & \cdot43^{9}\cdot47^{4}\cdot53^{3}\cdot59^{7}\cdot61^{9}\cdot67^{18}\cdot71^{12}\cdot73^{10}\cdot79^{6}\cdot83^{4}\cdot89^{2}\cdot97\cdot113\\ & \cdot127^{5}\cdot131^{7}\cdot137^{9}\cdot139^{10}\cdot149^{16}\cdot151^{17}\cdot157^{20}\cdot163^{24}\cdot167^{26}\cdot173^{30}\\ & \cdot179^{34}\cdot181^{36}\cdot191^{43}\cdot193^{44}\cdot197^{47}\cdot199^{47}\cdot211^{47}\cdot223^{44}\cdot\cdots\cdot9973. \end{aligned}$$

Note that the primes 101, 103, 107, and 109 are missing from the factorization. In other words they have exponent 0; of course, if that exponent had become negative then g_K would not be integral. The exponent of p in the factorization of g_K has an interesting self-similarity feature. Let

$$c_p(x)=x^{-1}\sum_{\ell=-\infty}^{\infty}p^{-\ell}||p^\ell x||^2$$

where $||x|| = \min_{n\in\mathbf{Z}} |n - x|$ is the distance to the nearest integer. Let $\nu_p(n)$ denote the highest power of p which divides n and let $[x]$ denote the greatest integer less than or equal to x. Then

$$\nu_p(g_{[p^j x]}) \sim [p^j x] c_p(x)$$

as $j \to \infty$.

12 Averages of Shifted Polynomials

Now we consider the more difficult problem of the autocorrelation of characteristic polynomials. Let

$$M_U(N, A, B) := \int_{U(N)} \prod_{\alpha\in A} \Lambda_X(e^{-\alpha}) \prod_{\beta\in B} \Lambda_{X^*}(e^{-\beta}) dX, \tag{93}$$

$$M_O(N, A) := \int_{O(2N)} \prod_{\alpha\in A} \Lambda_X(e^{-\alpha}) dX, \tag{94}$$

and

$$M_S(N, A) := \int_{USp(2N)} \prod_{\alpha\in A} \Lambda_X(e^{-\alpha}) dX. \tag{95}$$

12.1 Integrals of products of translates of characteristic polynomials

The results of Keating and Snaith give exact formulas for integrals of powers of characteristic polynomials. These lead to conjectures for the leading main terms of integrals of powers of the Riemann zeta-function on the critical line. However, in order to determine the lower order terms of the moments of the zeta-function it turns out to be necessary to take a different tack and to consider integrals of shifted characteristic polynomials.

Let $\Lambda_U(s) = \det(I - Us) = \prod_{h=1}^N (1 - se^{i\theta_h})$. Since the eigenvalues of U^* are the conjugares of the eigenvalues of U it follows that $\Lambda_{U^*}(s) = \prod_{h=1}^N (1 - se^{-i\theta_h})$.

We first illustrate the general result by stating the specific case when there are four characteristic polynomials:

$$\int_{U(N)} \Lambda_U(e^{-\alpha})\Lambda_U(e^{-\beta})\Lambda_{U^*}(e^{-\gamma})\Lambda_{U^*}(e^{-\delta})\ dU_N$$
$$= Z(\alpha,\beta,\gamma,\delta) + e^{-N(\alpha+\gamma)}Z(-\gamma,\beta,-\alpha,\delta) + e^{-N(\alpha+\delta)}Z(-\delta,\beta,\gamma,-\alpha)$$
$$+e^{-N(\beta+\gamma)}Z(\alpha,-\gamma,-\beta,\delta) + e^{-N(\beta+\delta)}Z(\alpha,-\delta,\gamma,-\beta) + e^{-N(\alpha+\beta+\gamma+\delta)}Z(-\gamma,-\delta,-\alpha,-\beta)$$

where

$$Z(\alpha,\beta,\gamma,\delta) = z(\alpha+\gamma)z(\alpha+\delta)z(\beta+\gamma)z(\gamma+\delta)$$

with $z(x) = 1/(1-e^{-x})$. Notice that there are 6 terms, each of which involves swapping some of the variables α, β with some of the variables γ, δ and introducing minus signs and factors in a way that depends on the "swapped" variables. The number 6, as we shall see, can be interpreted in two ways: one as $\binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2$ and the other as $\binom{4}{2}$.

In the general situation, we consider an integral

$$I(\alpha,\beta) := \int_{U(N)} \prod_{j=1}^{m} \Lambda_U(e^{-\alpha_j}) \prod_{k=1}^{n} \Lambda_{U^*}(e^{-\beta_k}) dU_N. \tag{96}$$

The answer is expressed in terms of the function

$$Z(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_n) = \prod_{j\le m, k\le n} z(\alpha_j+\beta_k)$$

and consists of a sum of $\binom{m+n}{m} = \sum_j \binom{m}{j}\binom{n}{j}$ terms. In each term, a set of α is exchanged with an equal-sized set of β, negative signs are introduced in all of the exchanged variables, and each term has a factor of $e^{-N\sum(\alpha+\beta)}$ where the sum is over the variables involved in the exchange.

We will phrase this result in two ways, one using set notation and the other using permutation notation. So, let V be the set of α and W be the set of β. Define

$$J_N(V,W) = \int_{U(N)} \prod_{\alpha\in V} \Lambda_U(e^{-\alpha}) \prod_{\beta\in W} \Lambda_{U^*}(e^{-\beta}) dU_N$$

and let

$$Z(V,W) = \prod_{\alpha\in V}\prod_{\beta\in W} z(\alpha+\beta)$$

with $z(x) = (1-e^{-x})^{-1}$. For a subset $v \subset V$ define $v^c = V - v$ and $v^- = \{-\alpha : \alpha \in v\}$, and similarly define w^c and w^-. Let $|w|$ denote the cardinality of w. Then our theorem is that

$$J_N(V,W) = \sum_{\substack{v\subset V, w\subset W\\ |v|=|w|}} e^{-N(\sum_{\alpha\in v}\alpha + \sum_{\beta\in w}\beta)} Z(v^c \cup w^-, v^- \cup w^c). \tag{97}$$

Now we reformulate this assertion in permutation notation. This restatement will be the version that we actually prove. To do this, it is most convenient to think of the set of α and the set of β as being sequences $(\alpha_1, \dots, \alpha_m)$ and $(\beta_1, \dots, \beta_n)$. It is further convenient to write $\beta_j = -\alpha_{m+j}$ so that the sequence $-\beta$ is appended onto the sequence α as $(\alpha_1, \dots, \alpha_m, -\alpha_{m+1}, \dots, -\alpha_{m+n})$. Then we are interested in computing

$$J_N(m,n) = \int_{U(N)} \prod_{j=1}^{m} \Lambda_U(e^{-\alpha_j}) \prod_{k=m+1}^{m+n} \Lambda_{U^*}(e^{\alpha_k}) dU_N.$$

Let $\Xi_{m,n}$ be defined as the set of permutations $\sigma \in \pi_{m+n}$ for which

$$\sigma(1) < \sigma(2) < \cdots < \sigma(m) \text{ and } \sigma(m+1) < \sigma(m+2) < \cdots < \sigma(m+n). \tag{98}$$

Given any such σ define a subset $S(\sigma)$ of $[1, m+n]$ of size m by $S(\sigma) = \{\sigma(j) : 1 \le j \le m\}$ and a set $T(\sigma)$ of size n by $T(\sigma) = \{\sigma(k) : m+1 \le k \le m+n\}$. Obviously, $T(\sigma) = [1, m+n] - S(\sigma)$. Also any subset S of $[1, m+n]$ of size m uniquely determines a complementary set T of size n and a $\sigma \in \Xi_{m,n}$ such that $S = S(\sigma)$ and $T = T(\sigma)$. Thus, the cardinality of $\Xi_{m,n}$ is $\binom{m+n}{m}$. Let $v(\sigma) = \{j \le m : \sigma(j) > m\}$ and let $w(\sigma) = \{k > m : \sigma(k) \le m\}$. Clearly, $|v(\sigma)| = |w(\sigma)|$ and these are just the sets of "swapped" variables v and w in the set theoretic formula (97) for $J_N(V,W)$. Note that any $\sigma \in \Xi_{m,n}$ uniquely determines the pair $(v(\sigma), w(\sigma))$ and vice versa, any pair (v,w) uniquely determines a $\sigma \in \Xi_{m,n}$. Observe that $v(\sigma)^c \cup w(\sigma) = S(\sigma)$ and $v(\sigma) \cup w(\sigma)^c = T(\sigma)$ (where the complement for v is with respect to $[1,m]$ and for w with respect to $[m+1, m+n]$) so that the Z factor in (97) is

$$Z(\sigma) = Z(S(\sigma), T(\sigma)) = \prod_{\substack{j\le m\\ m+1\le k\le m+n}} z(\alpha_{\sigma(j)} - \alpha_{\sigma(k)})$$

Observe also that

$$-\sum_{j\in v(\sigma)}\alpha_j+\sum_{k\in w(\sigma)}\alpha_k=\frac{1}{2}\Big(-\sum_{j=1}^{m}\alpha_j+\sum_{k=m+1}^{m+n}\alpha_k+\sum_{j=1}^{m}\alpha_{\sigma(j)}-\sum_{k=m+1}^{m+n}\alpha_{\sigma(k)}\Big)$$

so that the factor

$$e^{-N(\sum_{\alpha\in v}\alpha+\sum_{\beta\in w}\beta)}$$

from (97) is just

$$=e^{\frac{N}{2}(-\sum_{j=1}^{m}\alpha_j+\sum_{k=m+1}^{m+n}\alpha_k)}e^{\frac{N}{2}(\sum_{j=1}^{m}\alpha_{\sigma(j)}-\sum_{k=m+1}^{m+n}\alpha_{\sigma(k)})}.$$

Then the basic theorem can be phrased as

$$\int_{U(N)}\prod_{j=1}^{m}\Lambda_U(e^{-\alpha_j})\prod_{k=m+1}^{m+n}\Lambda_{U^*}(e^{\alpha_k})\;dU_N$$
$$=e^{\frac{N}{2}(-\sum_{j=1}^{m}\alpha_j+\sum_{k=m+1}^{m+n}\alpha_k)}\sum_{\sigma\in\Xi_{m,n}}e^{\frac{N}{2}(\sum_{j=1}^{m}\alpha_{\sigma(j)}-\sum_{k=m+1}^{m+n}\alpha_{\sigma(k)})}\prod_{\substack{j\le m\\ m+1\le k\le m+n}}z(\alpha_{\sigma(j)}-\alpha_{\sigma(k)}).$$

This is the version we will find convenient to prove.

By a combinatorial lemma, this can be expressed conveniently as

$$\int_{U(N)}\prod_{j=1}^{m}\Lambda_U(e^{-\alpha_j})\prod_{k=m+1}^{m+n}\Lambda_{U^*}(e^{\alpha_k})\;dU_N=\frac{(-1)^{m+n}}{m!n!}e^{-\frac{N}{2}(\sum_{j=1}^{m}\alpha_j-\sum_{k=m+1}^{m+n}\alpha_k)}\times$$
$$\int_{|w_1|>|\alpha|}\cdots\int_{|w_{m+n}|>|\alpha|}\frac{e^{\frac{N}{2}(\sum_{j=1}^{m}z_j-\sum_{k=m+1}^{m+n}z_k)}\Delta(w)^2\prod_{\substack{1\le j\le m\\ m+1\le k\le m+n}}z(w_j-w_k)}{\prod_{1\le j,k\le m+n}(w_j-\alpha_k)}\prod_{j=1}^{m+n}dw_j$$

In the case $m=n=k$ and the shifts are on the order of $O(1/N)$, the integral (96) is asymptotic to

$$\frac{1}{k!(2\pi i)^k}\int_{|w_1|=1}\cdots\int_{|w_k|=1}\frac{e^{N\sum_{j=1}^{k}w_i}\Delta(w)^2}{\prod_{h=1}^{k}\prod_{j=1}^{2k}(w_h-\alpha_j)}dw_1\dots dw_k$$

as $N\to\infty$.

12.2 Proof of shifted moment

Define

$$I_{m,n}(w)=\prod_{k=m+1}^{m+n} w_k^N \int_{U(N)} \prod_{j=1}^{m} \Lambda_{U^*}(w_j) \prod_{k=m+1}^{m+n} \Lambda_U(w_k^{-1})\ dU_N.$$

Now $w^N\Lambda_U(w^{-1})=\prod_{h=1}^N(w-e^{i\theta_h})$. Thus,

$$I_{m,n}(w)=\frac{1}{N!(2\pi)^N}\int_{[0,2\pi]^N}\prod_{\ell=1}^{N}\prod_{j=1}^{m}(1-e^{-i\theta_\ell}w_j)\prod_{k=m+1}^{m+n}(w_k-e^{i\theta_\ell})\prod_{1\le j<k\le N}|e^{i\theta_k}-e^{i\theta_j}|^2\, d\theta_1\dots d\theta_N$$

Lemma. *Suppose that a function $f(\theta_1,\dots,\theta_N)$ is symmetric in its N variables. Then*

$$\int_{[0,2\pi]^N} f(\theta_1,\dots,\theta_N)\prod_{1\le j<k\le N}|e^{i\theta_k}-e^{i\theta_j}|^2\ d\theta_1\dots d\theta_N$$
$$= N!\int_{[0,2\pi]^N} f(\theta_1,\dots,\theta_N)\prod_{1\le j<k\le N}(e^{i\theta_k}-e^{i\theta_j})\prod_{h=1}^{N} e^{-i(h-1)\theta_h}\ d\theta_1\dots d\theta_N$$

Proof. First of all

$$\begin{aligned}\prod_{j<k}|e^{i\theta_k}-e^{i\theta_j}|^2 &= \Delta(e^{i\theta_1},\dots,e^{i\theta_N})\Delta(e^{-i\theta_1},\dots,e^{-i\theta_N})\\ &= \Delta(e^{i\theta_1},\dots,e^{i\theta_N})\det_{N\times N}(e^{-i(j-1)\theta_k})\\ &= \Delta(e^{i\theta_1},\dots,e^{i\theta_N})\sum_{\sigma\in\pi_N}\operatorname{sgn}(\sigma)\prod_{h=1}^{N} e^{-i(h-1)\theta_{\sigma(h)}}\end{aligned}$$

Let $g(\theta)=g(\theta_1,\dots,\theta_N)=f(\theta_1,\dots,\theta_N)\Delta(\theta_1,\dots,\theta_N)$. Then

$$g(\theta_{\sigma 1},\dots,\theta_{\sigma N})=\operatorname{sgn}(\sigma)g(\theta_1,\dots,\theta_N)$$

and

$$\int_{[0,2\pi]^N} f(\theta_1,\ldots,\theta_N) \prod_{1\le j<k\le N} |e^{i\theta_k}-e^{i\theta_j}|^2 \; d\theta_1\ldots d\theta_N$$

$$= \int_{[0,2\pi]^N} g(\theta_1,\ldots,\theta_N) \sum_{\sigma\in\pi_N} \mathrm{sgn}(\sigma) \prod_{h=1}^{N} e^{-i(h-1)\theta_{\sigma(h)}} \; d\theta_1\ldots d\theta_N$$

$$= \sum_{\sigma\in\pi_N} \mathrm{sgn}(\sigma) \int_{[0,2\pi]^N} g(\theta_{\sigma^{-1}1},\ldots,\theta_{\sigma^{-1}N}) \prod_{h=1}^{N} e^{-i(h-1)\theta_h} \; d\theta_1\ldots d\theta_N$$

$$= \sum_{\sigma\in\pi_N} \int_{[0,2\pi]^N} g(\theta_1,\ldots,\theta_N) \prod_{h=1}^{N} e^{-i(h-1)\theta_h} \; d\theta_1\ldots d\theta_N$$

$$= N! \int_{[0,2\pi]^N} g(\theta_1,\ldots,\theta_N) \prod_{h=1}^{N} e^{-i(h-1)\theta_h} \; d\theta_1\ldots d\theta_N$$

as desired.

Returning to our proof of the shifted moments, we now have, after the Lemma,

$$I_{m,n}(w) = \frac{1}{(2\pi)^N} \int_{[0,2\pi]^N} G(w,\theta) \prod_{h=1}^{N} e^{-i(h-1)\theta_h} \; d\theta_1\ldots d\theta_N$$

where

$$\begin{aligned} G(w,\theta) &= \prod_{\ell=1}^{N} \left(\prod_{j=1}^{m} (1-e^{-i\theta_\ell} w_j) \prod_{k=m+1}^{m+n} (w_k - e^{i\theta_\ell}) \right) \prod_{1\le j<k\le N} (e^{i\theta_k}-e^{i\theta_j}) \\ &= \prod_{\ell=1}^{N} e^{-im\theta_\ell} \left(\prod_{j=1}^{m} (e^{i\theta_\ell} - w_j) \prod_{k=m+1}^{m+n} (w_k - e^{i\theta_\ell}) \right) \prod_{1\le j<k\le N} (e^{i\theta_k}-e^{i\theta_j}) \\ &= (-1)^{nN} \prod_{\ell=1}^{N} e^{-im\theta_\ell} \left(\prod_{j=1}^{m} (e^{i\theta_\ell} - w_j) \prod_{k=m+1}^{m+n} (e^{i\theta_\ell} - w_k) \right) \prod_{1\le j<k\le N} (e^{i\theta_k}-e^{i\theta_j}) \\ &= (-1)^{nN} \prod_{\ell=1}^{N} e^{-im\theta_\ell} \Delta(e^{i\theta_1},\ldots,e^{i\theta_N},w_1,\ldots,w_{m+n})/\Delta(w_1,\ldots,w_{m+n}) \end{aligned}$$

Thus,

$$I_{m,n}(w) = \frac{(-1)^{nN}(2\pi)^{-N}}{\Delta(w_1,\dots,w_{m+n})}\int_{[0,2\pi]^N}\prod_{\ell=1}^{N} e^{-i(m+\ell-1)\theta_\ell}\Delta(e^{i\theta_1},\dots,e^{i\theta_N},w_1,\dots,w_{m+n})\,d\theta_1\dots d\theta_N$$

We express $\Delta(e^{i\theta_1},\dots,w_{m+n})$ as a determinant with the jth row $(1,e^{i\theta_j},\dots,e^{i(N+m+n-1)\theta_j})$ for $1\le j\le N$ and $N+k$th row $(1,w_k,w_k^2,\dots,w_k^{N+m+n-1})$ for $1\le k\le m+n$. The factor $e^{-i(m+\ell-1)\theta_\ell}$ can be multiplied into the ℓth row. Now the variable θ_ℓ only appears in the ℓth row; thus, we can perform the integral over θ_ℓ by integrating each entry of the ℓth row with respect to θ_ℓ. The integration produces a 0 in each entry of the ℓth row except for a 2π in the $m+\ell$ position. We can expand the resulting determinant along these rows of mostly zeros and obtain that

$$I_{m,n}(w) = \frac{\det_{m+n}\left(w_j^{k-1+N\delta(k)}\right)}{\det_{m+n}\left(w_j^{k-1}\right)}$$

where $\delta(k)=0$ if $k\le m$ and $\delta(k)=1$ if $m+1\le k\le m+n$. Thus, we recognize $I_{m,n}$ as a Schur polynomial associated with the partition $(0,\dots,0,N,\dots,N)$; the number of zeros is m and the number of Ns is n.

We expand the determinant in the numerator into a sum over permutations $\eta\in\pi_{m+n}$

$$\det_{m+n}\left(w_j^{k-1+N\delta(k)}\right) = \sum_{\eta\in\pi_{m+n}}\operatorname{sgn}(\eta)\prod_{j=1}^{m+n} w_{\eta(j)}^{j-1+N\delta(j)}.$$

Any $\eta\in\pi_{m+n}$ can be expressed uniquely as $\eta=\tau\rho\sigma$ where $\sigma\in\Xi_{m,n}$ and where ρ is a permutation of $\{\sigma(1),\dots,\sigma(m)\}$ and where τ is a permutation of $\{\sigma(m+1),\dots,\sigma(m+n)\}$. Thus,

$$\begin{aligned}\det_{m+n}\left(w_j^{k-1+N\delta(k)}\right) &= \sum_{\sigma\in\Xi_{m,n}}\operatorname{sgn}(\sigma)\sum_\rho\operatorname{sgn}(\rho)\prod_{j=1}^{m} w_{\rho\sigma(j)}^{j-1}\sum_\tau\operatorname{sgn}(\tau)\prod_{k=m+1}^{m+n} w_{\tau\sigma(k)}^{k-1}\prod_{k=m+1}^{m+n} w_{\sigma(k)}^{N}\\ &= \sum_{\sigma\in\Xi_{m,n}}\Delta(w_{\sigma(1)},\dots,w_{\sigma(m)})\Delta(w_{\sigma(m+1)},\dots,w_{\sigma(m+n)})\prod_{k=m+1}^{m+n} w_{\sigma(k)}^{m+N}\end{aligned}$$

Therefore

$$I_{m,n}(w) = (-1)^{nN}\sum_{\sigma\in\Xi_{m,n}}\frac{\prod_{k=m+1}^{m+n} w_{\sigma k}^N}{\prod_{\substack{1\le j\le m\\ m+1\le k\le m+n}}(1-w_{\sigma(j)}w_{\sigma(k)}^{-1})} \tag{99}$$

Now taking $w_j=e^{-\alpha_j}$ we obtain the result claimed.

12.3 A product formula

We evaluate the limit as all of the $w_j \to 1$ of the shifted moment and arrive at the product formula found previously from Selberg's integral. Returning to

$$I_{m,n}(w) = \frac{\det_{m+n}\left(w_j^{k-1+N\delta(k)}\right)}{\det_{m+n}\left(w_j^{k-1}\right)}$$

we first of all set $w_1 = 1$. This gives a first row of all ones in the numerator. Then we subtract the first row from the second and divide by the factor $w_2 - 1$ in the denominator and take the limit as $w_2 \to 1$. This gives a second row which is $0, 1, 2, \ldots, m-1, m+N, \ldots m+n-1+N$. Now the denominator has a factor of $(w_3-1)^2$. So we subtract the first row and (w_3-1) times the second row from the third row, divide by $(w_3-1)^2$ and take the limit as $w_3 \to 1$. This produces a third row which is $0, 0, 1, 3, 6, \ldots, \binom{m-1}{2}, \binom{m+N}{2}, \ldots, \binom{m+n-1+N}{2}$. Now the denominator has a factor of $(w_4-1)^3$. So, from the fourth row we subtract the first row plus (w_4-1) times the second row plus $(w_4-1)^2$ times the third row. When we divide by $(w_4-1)^3$ and take the limit as $w_4 \to 1$ the fourth row has entries $\binom{k-1+N\delta(k)}{3}$ where $\delta(k) = 0$ if $k \le m$ and $= 1$ if $m+1 \le k \le m+n$. We continue this procedure and conclude that

$$I_{m,n}(1) \quad = \quad \det_{m+n}\binom{k-1+N\delta(k)}{j-1}.$$

Now the upper left $m \times m$ square has the property that the entries below the main diagonal are 0 and on the main diagonal are 1. Hence, the determinant reduces to an $n \times n$ determinant formed from the lower right $n \times n$ square. Thus,

$$I_{m,n}(1) = \det_n \binom{m+k-1+N}{m+j-1}.$$

(This formula verifies one of the claims made earlier about $g(k, N)$.) Note that $\binom{m+k-1+N}{m+j-1} = \frac{(N+m+k-1)!}{(m+j-1)!(N+k-j)!}$. To recover the product formula, we factor $(N+m+k-1)!/(N+k-1)!$ out of each column and $1/(m+j-1)!$ out of each row. Then we see that

$$I_{m,n}(1) = \prod_{h=1}^{n} \frac{(N+M+h-1)!}{(N+h-1)!(m+h-1)!} D(n, N)$$

where

$$D(n,N) = \det_{n\times n}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ N & N+1 & N+2 & \dots & N+n-1 \\ N(N-1) & (N+1)N & (N+2)(N+1) & \dots & \\ N(N-1)(N-2) & & & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ N(N-1)\dots(N-n+2) & & & & \end{pmatrix}$$

This determinant can be evaluated by an induction argument. If we subtract the jth column from the $(j+1)$st column we see that

$$D(n,N) = \det_{n\times n}\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ N & 1 & 1 & \dots & 1 \\ N(N-1) & 2N & 2(N+1) & \dots & 2(N+n-2) \\ N(N-1)(N-2) & 2N(N-1) & & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

which is equal to $(n-1)!D(n-1,N)$. Thus, $D(n,N) = 1!\,2!\dots(n-1)!$ and

$$\begin{aligned} I_{m,n}(1) &= \prod_{h=1}^{n} \frac{(N+m+h-1)!(h-1)!}{(N+h-1)!(m+h-1)!} \\ &= \frac{G(N+m+n+1)G(N+1)}{G(N+m+1)G(N+n+1)} \frac{G(m+1)G(n+1)}{G(m+n+1)}. \end{aligned}$$

12.4 Combinatorial Lemma

Each term in the sum over $\sigma \in \Xi_{m,n}$ has a pole of order mn at $(0,\dots,0)$. At first sight, it seems somewhat remarkable that all of these poles cancel out when the sum over all such permutations is taken. The best way we have found to see that this is indeed the case is to re-express this sum as the residue of a function which is analytic apart from its poles near $(0,\dots,0$. The point, then, is that the sum can be expressed as an integral over a path that encloses these poles but does not touch them; this renders the sum visibly analytic in the neighborhood of $(0,\dots,0)$.

Here is a general lemma which expresses these ideas.

Lemma. *Suppose that* $F(a;b) = F(a_1,\dots,a_m;b_1,\dots,b_n)$ *is symmetric in the a variables and in the b variables and is regular near* $(0,\dots,0)$*. Suppose* $f(s) = \frac{1}{s} + c + \dots$ *and let*

$$G(a_1,\dots,a_m;b_1,\dots b_n) = F(a_1,\dots;\dots,b_n)\prod_{i=1}^{m}\prod_{j=1}^{n} f(a_i - b_j).$$

Let $\Xi_{m,n}$ *be as defined in (98). Then*

$$\sum_{\sigma\in\Xi_{m,n}} G(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(m)};\alpha_{\sigma(m+1)}\dots\alpha_{\sigma(m+n)}) =$$

$$\frac{(-1)^{(m+n)}}{m!n!}\sum_{\sigma\in\pi_{m+n}} \mathrm{Res}_{(z_1,\dots,z_{m+n})=(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(m+n)})}\frac{G(z_1,\dots,z_{m+n})\Delta(z_1,\dots,z_{m+n})^2}{\prod_{i=1}^{m+n}\prod_{j=1}^{m+n}(z_i-\alpha_j)},$$

Proof. It suffices to prove that

$$\mathrm{Res}_{(z_1,\dots,z_{m+n})=(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(m+n)})}\frac{\Delta(z_1,\dots,z_{m+n})^2}{\prod_{i=1}^{m+n}\prod_{j=1}^{m+n}(z_i-\alpha_j)} = (-1)^{m+n}$$

since each such term will appear $m!n!$ times. Consider the case where σ is the identity permutation. Then the residue is

$$= \frac{\prod_{j<k}(\alpha_k-\alpha_j)^2}{\prod_{j\neq k}(\alpha_j-\alpha_k)} = (-1)^{m+n};$$

the answer will be the same for any permutation σ.

The residue (100) can be expressed as $(2\pi i)^{-m-n}$ times an $m+n$ fold integral, each path of which encircles all of the poles of the integrand; note that the value of such an integral may be calculated by summing the residues and note that there is no singularity when $z_j = z_k$ because of the factor $(z_k - z_j)^2$ in the numerator.

12.5 Symplectic averages

Let

$$A = \{\alpha_1,\dots,\alpha_M\}. \tag{100}$$

It is convenient to let

$$\alpha_m = -ia_m. \tag{101}$$

Then for a matrix $X \in USp(2N)$ with eigenvalues $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N}$ a small calculation shows that

$$\Lambda_X(e^{ia}) = (2e^{ia})^N \prod_{n=1}^{N} (\cos a - \cos\theta_n). \tag{102}$$

Therefore,

$$M_{S,N}(A) = \frac{2^{N^2}}{\pi^N N!} \int_{[0,\pi]^N} \prod_{m=1}^{M} (2e^{ia_m})^N \prod_{n=1}^{N} (\cos a_m - \cos\theta_n) \tag{103}$$
$$\times \prod_{1\le j<k\le N} (\cos\theta_k - \cos\theta_j)^2 \prod_{n=1}^{N} \sin^2\theta_n \; d\theta_1 \ldots d\theta_N.$$

We rewrite the integrand in such a way that we have a product of two Vandermondes $\Delta(\cos\theta_1, \ldots, \cos\theta_N, \cos a_1, \ldots, \cos a_M)\Delta(\cos\theta_1, \ldots, \cos\theta_N)$:

$$M_{S,N}(A) = \frac{2^{N^2+MN} \exp(iN\sum_{m=1}^{M} a_m)}{\pi^N N! \Delta(\cos a_1, \ldots, \cos a_M)} \int_{[0,\pi]^N} \Delta(\cos\theta_1, \ldots, \cos\theta_N)$$
$$\times \Delta(\cos\theta_1, \ldots, \cos\theta_N, \cos a_1, \ldots, \cos a_M) \prod_{n=1}^{N} \sin^2\theta_n \; d\theta_1 \ldots d\theta_N.$$

We now use (15) to rewrite the Vandermondes as determinants with the Chebyshev U polynomials. It is convenient to denote a_m by θ_{N+m}. Thus,

$$M_{S,N}(A) = \frac{2^{N-\frac{M(M+1)}{2}-2} \exp(iN\sum_{m=1}^{M} \theta_{N+m})}{\pi^N N! \det_{M\times M}(U_{j-1}(\cos a_k))} \int_{[0,\pi]^N} \det_{N\times N}(U_{j-1}(\cos\theta_k))$$
$$\times \det_{N+M\times N+M}(U_{j-1}(\cos\theta_k)) \prod_{n=1}^{N} \sin^2\theta_n \; d\theta_1 \ldots d\theta_N.$$

Now we use Andréief's generalized identity (24) to perform the integration, much as when we verified that the mass of the Haar measure is 1. In this way we obtain

$$\int_{[0,\pi]^N} \det_{N\times N}(U_{j-1}(\cos\theta_k)) \det_{N+M\times N+M}(U_{j-1}(\cos\theta_k)) \prod_{n=1}^{N} \sin^2\theta_n \; d\theta_1 \dots d\theta_N$$
$$= \det_{N+M} \left(\mathcal{T}_j U_{j-1}(\cos\theta) U_{k-1}(\cos\theta)\right)$$

where $\mathcal{T}_j$ is integration with respect to the measure $\sin^2\theta d\theta$ if $j \le N$ and is evaluation at $\theta_j = a_{j-N}$ if $j > N$. By the orthogonality of the U_j, the first N rows of the determinant are 1 on the diagonal and 0 elsewhere. Thus, the determinant looks like

$$\det \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 \\ 1 & U_1(\cos a_1) & \dots & U_{N-1}(\cos a_1) & \dots & U_{N+M-1}(\cos a_1) \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 1 & U_1(\cos a_M) & \dots & U_{N-1}(\cos a_M) & \dots & U_{N+M-1}(\cos a_M) \end{pmatrix}$$

Thus,

$$M_{S,N}(A) \;=\; 2^{-\frac{M(M+1)}{2}-2} e^{iN\sum_{m=1}^{M} a_m} \frac{\det_{M\times M}(U_{N+j-1}(\cos a_k))}{\Delta(\cos a_1, \dots, \cos a_M)}. \quad (104)$$

Recall that

$$U_{j-1}(\cos\theta)) = \frac{\sin j\theta}{\sin\theta}.$$

Thus,

$$M_{S,N}(A) \;=\; \frac{2^{-\frac{M(M+1)}{2}-2} e^{iN\sum_{m=1}^{M} a_m}}{\prod_{m=1}^{M} \sin a_m} \frac{\det_{M\times M}(\sin(N+j)a_k)}{\Delta(\cos a_1, \dots \cos a_M)}.$$

Now we return to $\alpha_m = -ia_m$, using $\cos ia = \cosh a$ and $\sin ia = i\sinh a$. Then,

$$M_{S,N}(A) \;=\; \frac{(-1)^M 2^{-\frac{M(M+1)}{2}-2} e^{-N\sum_{m=1}^{M} \alpha_m}}{\prod_{m=1}^{M} \sinh \alpha_m} \frac{\det_{M\times M}(\sinh(N+j)\alpha_k)}{\Delta(\cosh\alpha_1, \dots \cosh\alpha_M)}.$$

To put this last expression into a particular shape that we desire, we write $\sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$ for each sinh function in the determinant. Then we split up the determinant column-by-column repeatedly using

$$\det\begin{pmatrix} b_{1,1}+c_{1,1} & b_{1,2} & \dots & b_{1,M} \\ b_{2,1}+c_{2,1} & b_{2,2} & \dots & b_{2,M} \\ \vdots & \vdots & \dots & \vdots \\ b_{M,1}+c_{M,1} & b_{M,2} & \dots & b_{M,M} \end{pmatrix} =$$

$$\det\begin{pmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,M} \\ b_{2,1} & b_{2,2} & \dots & b_{2,M} \\ \vdots & \vdots & \dots & \vdots \\ b_{M,1} & b_{M,2} & \dots & b_{M,M} \end{pmatrix} + \det\begin{pmatrix} c_{1,1} & b_{1,2} & \dots & b_{1,M} \\ c_{2,1} & b_{2,2} & \dots & b_{2,M} \\ \vdots & \vdots & \dots & \vdots \\ c_{M,1} & b_{M,2} & \dots & b_{M,M} \end{pmatrix}$$

This leads to

$$\begin{aligned} \det_{M\times M}(\sinh(N+j)\alpha_k) &= 2^{-M} \sum_{\substack{1\le m\le M \\ \epsilon_m\in\{-1,+1\}}} (\prod_{m=1}^{M} \epsilon_m) e^{(N+1)\sum_{m=1}^{M}\epsilon_m\alpha_m} \det(e^{\epsilon_j(j-1)\alpha_k}) \\ &= 2^{-M} \sum_{\substack{1\le m\le M \\ \epsilon_m\in\{-1,+1\}}} (\prod_{m=1}^{M} \epsilon_m) e^{(N+1)\sum_{m=1}^{M}\epsilon_m\alpha_m} \Delta(e^{\epsilon_1\alpha_1},\dots,e^{\epsilon_M\alpha_M}). \end{aligned}$$

Now we observe that

$$\frac{\Delta(e^{\epsilon_1\alpha_1},\dots,e^{\epsilon_M\alpha_M})}{\Delta(\cosh\alpha_1,\dots\cosh\alpha_M)} = 2^{\frac{M(M-1)}{2}} \prod_{1\le j<k\le M} z(\epsilon_j\alpha_j+\epsilon_m\alpha_m) \qquad (105)$$

where, as usual,

$$z(x) = \frac{1}{1-e^{-x}}.$$

Inserting these last two expressions into (104) we have

$$\begin{aligned} M_{S,N}(A) &= \frac{(-1)^M 2^{-\frac{M(M+1)}{2}-2} e^{-N\sum_{m=1}^{M}\alpha_m}}{\prod_{m=1}^{M}\sinh\alpha_m} 2^{-M} \\ &\qquad \sum_{\substack{1\le m\le M \\ \epsilon_m\in\{-1,+1\}}} (\prod_{m=1}^{M} \epsilon_m) e^{(N+1)\sum_{m=1}^{M}\epsilon_m\alpha_m} 2^{\frac{M(M-1)}{2}} \prod_{1\le j<k\le M} z(\epsilon_j\alpha_j+\epsilon_m\alpha_m). \end{aligned}$$

Finally, we use

$$\frac{\epsilon e^{\epsilon\alpha}}{\sinh \epsilon\alpha} = 2z(2\epsilon\alpha), \tag{106}$$

valid for $\epsilon = \pm 1$, to obtain

Theorem 1

$$M_{S,N}(A) = \sum_{\substack{1\le m\le M\\ \epsilon_m\in\{-1,+1\}}} e^{N\sum_{m=1}^{M}(\epsilon_m\alpha_m-\alpha_m)} \prod_{1\le j\le k\le M} z(\epsilon_j\alpha_j+\epsilon_m\alpha_m).$$

For example,

$$\int_{USp(2N)} \Lambda_X(e^{-\alpha})dX = z(2\alpha) + e^{-2\alpha N}z(-2\alpha)$$

and

$$\begin{aligned}\int_{USp(2N)} \Lambda_X(e^{-\alpha})\Lambda_X(e^{-\beta})dX &= z(2\alpha)z(\alpha+\beta)z(2\beta)\\ &\quad + e^{-2\alpha N}z(-2\alpha)z(-\alpha+\beta)z(2\beta)\\ &\quad + e^{-2\beta N}z(2\alpha)z(\alpha-\beta)z(-2\beta)\\ &\quad + e^{-2(\alpha+\beta)N}z(-2\alpha)z(-\alpha-\beta)z(-2\beta).\end{aligned}$$

13 Averages of ratios for unitary ensembles

13.1 Toeplitz determinants

Matrices of the form (a_{j-k}) are called Toeplitz matrices. Their entries are constant on diagonals. We are interested in their determinants in the special case that the entries a_m are generated as the Laurent coefficients of a function f. This situation arises in the Gram's identity:

$$\frac{1}{N!}\int_{S^N}\prod_{i=1}^{N} f(\theta_i) \det_{N\times N}(\phi_j(\theta_k)) \det_{N\times N}(\psi_j(\theta_k))\, d\theta_1\dots d\theta_N = \det_{N\times N}\Big(\int_S f(\theta)\phi_j(\theta)\overline{\psi_k(\theta)}\, d\theta\Big)$$

For example,

$$\int_{U(N)} \prod_{h=1}^{N} f(e^{i\theta_h}) dU_N = \frac{1}{(2\pi)^N} \det_{N\times N} \left(\int_0^{2\pi} f(e^{i\theta}) e^{i(j-k)\theta} \, d\theta \right) \tag{107}$$

and the determinant on the right is a Toeplitz determinant. This identity is also called Heine's identity.

Suppose that f has a Laurent series expansion around $z = 0$:

$$f(z) = \sum_{n=-H}^{\infty} \gamma_n z^n.$$

Then

$$\int_0^{2\pi} f(e^{i\theta}) e^{-im\theta} \, d\theta = \frac{1}{i} \int_{|z|=1} \frac{f(z)}{z^{m+1}} dz = 2\pi \gamma_m.$$

Thus,

$$\int_{U(N)} \prod_{h=1}^{N} f(e^{i\theta_h}) dU_N = \det_{N\times N} (a_{j-k}) \tag{108}$$

K. M. Day found a formula for the evaluation of such a Toeplitz determinant in the case that $f(t)$ is a rational function

$$f(t) = \frac{\prod_{h=1}^{p} (t - r_h)}{\prod_{h=1}^{q} (1 - \frac{t}{u_h}) \prod_{h=1}^{s} (t - v_h)}$$

where the r_h are distinct, and $|u_h| > 1, |v_h| < 1$ and $p \geq s$. Day's formula, which is valid when $N \geq s - 1$, is that

$$\det_{N\times N} (a_{j-k}) = \sum_{\substack{M \subset [1,p] \\ |M|=s}} C_M (r_M)^N$$

where

$$r_M = (-1)^{p-s} \prod_{j \in \overline{M}} r_j$$

and

$$C_M = \frac{\prod_{\substack{j \in \overline{M} \\ \ell \leq s}} (r_j - v_\ell) \prod_{\substack{k \in M \\ i \leq q}} (u_i - r_k)}{\prod_{\substack{\ell \leq s \\ i \leq q}} (u_i - v_\ell) \prod_{\substack{j \in \overline{M} \\ k \in M}} (r_j - r_k)}.$$

The amazing thing is the simple dependence on N (when $N \geq s-1$). Day's formula can be used to give an alternate evaluation of integrals of products of shifted characteristic polynomials, since

$$\Lambda_U(b^{-1}) = \prod_{h=1}^{N}(1 - e^{i\theta_h}/b) = \prod_{h=1}^{N} f(e^{i\theta_h})$$

with $f(t) = (1 - t/b)$ and

$$\Lambda_{U^*}(a) = \prod_{h=1}^{N}(1 - ae^{-i\theta_h}) = \prod_{h=1}^{N} f(e^{i\theta_h})$$

with $f(t) = (1-b/t)$. Thus, using our earlier notation with $w = (w_1, \dots, w_{m+n}) = (a_1, \dots, a_m, b_1, \dots, b_n)$ we have

$$I_{m,n}(w) = \int_{U(N)} \prod_{j=1}^{m} \Lambda_{U^*}(a_j) \prod_{k=1}^{n} b_k^N \Lambda_U(1/b_k) dU_N = \det_{N\times N}(\gamma_{j-k})$$

where the γ_m are the Laurent coefficients of

$$f(t) = \prod_{j=1}^{m}(1 - a_j/t) \prod_{k=1}^{n} b_k(1 - t/b_k) = (-1)^n t^{-m} \prod_{j=1}^{m}(t - a_j) \prod_{k=1}^{n}(t - b_k)$$

Recall that we earlier proved that

$$I_{m,n}(w) = \sum_{\sigma\in\Xi_{m,n}} \frac{\prod_{k=m+1}^{m+n} w_{\sigma k}^N}{\prod_{\substack{1\leq j\leq m\\ m+1\leq k\leq m+n}}(1 - w_{\sigma(j)} w_{\sigma(k)}^{-1})}.$$

To relate this determinant to Day's formula, we take $s = m$, $q = 0$, $p = m + n$, all of the $v_\ell = 0$ and the r_j are the union of the a_j and the b_k. Day's formula evaluates the Toeplitz determinant as a sum over subsets M of $[1, m+n]$ of size m and so there are $\binom{m+n}{n}$ terms in the sum, just as we found in our previous evaluation. A subset M of $[1, m+n]$ of size m corresponds uniquely to a permutation $\sigma \in \pi_{m+n}$ with $\sigma(1) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(m+n)$ (hence an element of $\Xi_{m,n}$). The identification is just $M = \{\sigma(1), \dots, \sigma(m)\}$. Thus, for example, in Day's formula, $\prod_{j\in M} r_j$ translates to $\prod_{j=1}^m r_{\sigma(j)}$. In this notation, Day's result is

$$I_{m,n}(w) = (-1)^{nN} \frac{\prod_{k=1}^{n} w_{\sigma(k)}^{n+N}}{\prod_{\substack{1\leq j\leq m\\ 1\, ek\leq n}}(w_{\sigma(k)} - w_{\sigma(j)})} = (-1)^{nN} \frac{\prod_{k=1}^{n} w_{\sigma(k)}^{N}}{\prod_{\substack{1\leq j\leq m\\ 1\, ek\leq n}}(1 - w_{\sigma(k)}^{-1} w_{\sigma(j)})} \tag{109}$$

as before.

Day's proof is elementary, but somewhat complicated. Basor and Forrester found another method to evaluate such a determinant formula, which is simpler to derive and can be shown to agree with Day's formula in the case that $q = 0$, which is the case in (109).

We now give a sketch of the proof of Basor and Forrester, which is stated in terms of the function

$$g(t) = \frac{\prod_{j=1}^{p}(t - r_j)}{\prod_{j=1}^{q}(1/t - u_j)\prod_{j=1}^{s}(t - v_j)}.$$

Let $t_k = e^{i\theta_k}$ and directly consider

$$\int_{[0,2\pi]^N} \prod_{k=1}^{N} g(t_k)\Delta(t_1, \dots, t_N)\Delta(1/t_1, \dots, 1/t_N)\ d\theta_1 \dots d\theta_N.$$

Let

$$D(a_1, \dots, a_M; b_1, \dots, b_N) = \prod_{\substack{1\le j\le N\\ 1\le k\le N}} (b_k - a_j).$$

Then the integrand is

$$\frac{D(r;t)\Delta(t)\Delta(1/t)}{D(u;1/t)D(v;t)}$$

where r stands for the sequence $(r_1, \dots, r_p)$, t stands for $(t_1, \dots, t_N)$, $1/t$ stands for $(1/t_1, \dots, 1/t_N)$, and so on. Observe that if $(a \sqcup b)$ is the concatenation of the sequences a and b, then $\Delta(a \sqcup b) = \Delta(a)D(a;b)\Delta(b)$. Also, $D(a \sqcup b; c) = D(a;c)D(b;c)$.

Recall the formula for the Vandermonde determinant:

$$\det_{M\times M}(a_j^{k-1}) = \Delta(a) := \prod_{1\le j<k\le M} (a_k - a_j) \tag{110}$$

for a sequence $a = (a_1, \dots, a_M)$ and the Cauchy double alternant formula:

$$\det_{N\times N}\left(\frac{1}{a_j - b_k}\right) = (-1)^{\frac{N(N+1)}{2}}\frac{\Delta(a)\Delta(b)}{D(a;b)} \tag{111}$$

for sequences $a = (a_1, \dots, a_N)$ and $b = (b_1, \dots, b_N)$. (The Cauchy formula follows by considering the degree of the determinant together with its obvious poles and zeros, just as the proof of the product formula for the Vandermonde; to determine the correct sign consider the coefficient of $a_2 a_3^2 \ldots a_M^{M-1} b_2 b_3^2 \ldots b_M^{M-1}$ which arises only from the term $\prod_{j=1}^M (a_j - b_j)^{-1}$.) Basor and Forrester proved the following hybrid formula which includes both of these results. Given sequences $a = (a_1, \ldots, a_M)$ and $b = (b_1, \ldots, b_N)$ with $N \le M$, we have

$$\det_{M\times M}\left(a_j^{k-1} \,\middle|\, \frac{1}{a_j - b_k}\right) = (-1)^{MN} \frac{\Delta(a)\Delta(b)}{D(a;b)} = \frac{\Delta(a)\Delta(b)}{D(b;a)};$$

the matrix (a_j^{k-1}) on the left is size $M \times (M-N)$ and the matrix $(1/(a_j - b_k))$ on the right is of size $M \times N$. This formula also follows by comparison of degrees and poles and zeros. Let us abbreviate the matrix here as $(V(a) \mid C(a;b))$, with V for Vandermonde and C for Cauchy.

Now it is an easy matter to check from these formulas that the integrand is

$$\frac{D(r;t)\Delta(t)\Delta(1/t)}{D(u;1/t)D(v;t)} = \frac{(-1)^{N(p+q)} D(v;r)}{\Delta(r)\Delta(v)\Delta(u)} \det_{N+p}(V(t \sqcup r) \mid C(t \sqcup r; v)) \det_N (V(1/t) \mid C(u;1/t))$$

Note that the jth row of $(V(t \sqcup r) \mid C(t \sqcup r; v))$ is

$$1 \qquad t_j \qquad \ldots \qquad t_j^{N+p-s-1} \qquad \frac{1}{v_s - t_j} \qquad \ldots \qquad \frac{1}{v_1 - t_j}$$

if $j \le N$ and is

$$1 \qquad r_{j-N} \qquad \ldots \qquad r_{j-N}^{N+p-s-1} \qquad \frac{1}{v_s - r_{j-N}} \qquad \ldots \qquad \frac{1}{v_1 - r_{j-N}}$$

if $j > N$. Also, the jth row of $(V(1/t) \mid C(u;1/t))$ is

$$1 \qquad t_j^{-1} \qquad \ldots \qquad t_j^{-(N-q-1)} \qquad \frac{1}{u_q - t_j^{-1}} \qquad \ldots \qquad \frac{1}{u_1 - t_j^{-1}}.$$

Next, we expand $\det_N(V(1/t)|C(u;1/t))$ into a sum over permutations

$$\det_N(V(1/t) \mid C(u;1/t)) = \sum_{\sigma \in \pi_N} \operatorname{sgn}(\sigma) \prod_{j=1}^N f_{\sigma(j)}(t_j)$$

where $f_j(t) = t^{-j-1}$ if $j \le N - q$ and $f_j(t) = 1/(u_k - 1/t)$ if $j = N - k + 1$ with $k \ge 1$. We then multiply the factor $f_{\sigma(j)}(t_j)$ into the jth row of the other determinant, when $j \le N$. This produces a determinant where the jth row for $j \le N$ depends only on t_j. Therefore, we can bring the integration over t_j into the jth row of that determinant. (The above argument is just a slight variation of the first Gram formula.)

To perform the integrations we use the formulae

$$\int_0^{2\pi} \frac{e^{-ij\theta}}{u - e^{i\theta}}\, d\theta = 0, \qquad (j = 0, 1, 2, \dots)$$

and

$$\int_0^{2\pi} \frac{1}{(u - e^{-i\theta})(v - e^{i\theta})}\, d\theta = \frac{1}{1 - uv}.$$

When we perform the integrations we get 0 or 1 for each entry in the top $N - q$ rows.

We deduce that the integral is

$$\frac{(-1)^{N(p+q)} D(v; r)}{\Delta(r)\Delta(v)\Delta(u)} \det_{p+q} H(r, u, v)$$

where the jth row of $H(r, u, v)$ is

$$0 \quad 0 \quad \dots \quad 0 \quad (1 - u_{q-j+1}v_s)^{-1} \quad \dots \quad (1 - u_{q-j+1}v_1)^{-1}$$

if $j \le q$ and is

$$r_{j+1-q}^{n+q} \quad r_{j+1-q}^{n+q+1} \quad \dots \quad r_{j+1-q}^{n+s-p-1} \quad (v_s - r_{j+1-q})^{-1} \quad \dots \quad (v_1 - r_{j+1-q})^{-1}$$

if $j > q$.

We evaluate the determinant via a Laplace expansion – not the usual one which involves expanding along one row or column – but the version that allows for the evaluation of an $M \times M$ determinant by expanding along a set S of n columns and produces a linear combination that is an alternating sum of $\binom{M}{n}$ products of an $n \times n$ determinant (taken by choosing n rows from the set S of columns) and multiplying by the $(M - n) \times (M - n)$ minor.

Our matrix has a $q \times (p + q - s)$ array of zeros in the upper left. So, we expand down the first $p + q - s$ columns – these columns with initial 0s.

We have to choose all sets of $p+q-s$ rows from all of the $p+q$ rows and multiply a $(p+q-s)\times(p+q-s)$ determinant by an $s\times s$ determinant. If one of the rows selected is one of the initial rows of zeros, we of course obtain a contribution of 0. Therefore, we are really choosing $p+q-s$ rows from the bottom p rows. We see that if $q>s$ then the determinant is 0, so we assume that $q\le s$.

Because of this initial rectangle of 0s, we let σ be a permutation in $\Xi_{p+q-s,s}$ so that with $\sigma(1)<\sigma(2)<\cdots<\sigma(p+q-s)$ and $\sigma(p+q-s+1)<\cdots<\sigma(p)$. Then, we have a sum of products of two determinants. The first determinant is a Vandermonde determinant $V(r_{\sigma(1)},\dots,r_{\sigma(p+q-s)})$ and the second is $\prod_{j=1}^q(-u_j)^{-1}$ times a Cauchy determinant $C(v_j;u_j^{-1}\sqcup r_{\sigma(p+q-s+j)})$. Each of these determinants may be expressed as a product as noted in (110) and (111). Thus we obtain a formula for the integral:

$$(-1)^{N(p+q)}\sum_{\sigma\in\Xi_{p+q-s,s}}\operatorname{sgn}(\sigma)\frac{D(v;r)\Delta(r_\sigma)\prod_{j=1}^q(-u_j)^{-1}\Delta(v)\Delta(u^{-1}\sqcup r_{\overline{\sigma}})}{\Delta(r)\Delta(v)\Delta(u)D(v;u^{-1}\sqcup r_{\overline{\sigma}})} \tag{112}$$

where $r_\sigma=(r_{\sigma(1)},\dots,r_{\sigma(p+q-s)})$ and $r_{\overline{\sigma}}=(r_{\sigma(p+q-s+1)},\dots,r_{\sigma(p)}$. Now

$$\Delta((u^{-1}\sqcup r_{\overline{\sigma}})=\Delta(u^{-1})\Delta(r_{\overline{\sigma}})D(u^{-1},r_{\overline{\sigma}})$$

and

$$D(v;u^{-1}\sqcup r_{\overline{\sigma}})=D(v,u^{-1})D(v,r_{\overline{\sigma}}).$$

Note also that $\Delta(u^{-1})=\Delta(u)/\prod_{j=1}^q u_j^{-(q-1)}$ and $\Delta(r)=\Delta(r_\sigma)\Delta(r_{\overline{\sigma}})D(r_\sigma,r_{\overline{\sigma}})$. Thus, (112) simplifies to

$$\frac{(-1)^{Np+(N+1)q}}{D(u,v)\prod_{j=1}^q u_j^q}\sum_{\sigma\in\Xi_{p-s,s}}\operatorname{sgn}(\sigma)\frac{D(v;r_\sigma)D(u^{-1},r_{\overline{\sigma}})}{D(r_\sigma,r_{\overline{\sigma}})}$$

13.2 Integrals of ratios of characteristic polynomials

We apply Day's formula to evaluate

$$\int_{U(N)}\frac{\prod_{j=1}^m\Lambda_{U^*}(w_j)w_{m+j}^N\Lambda_U(w_{m+j}^{-1})}{\prod_{j=1}^m\Lambda_{U^*}(y_j)\Lambda_U(z_j^{-1})}dU_N.$$

where $|y_j| < 1$, $|z_j| > 1$ and the w_j are distinct. This is equivalent to evaluating the Toeplitz determinant with rational function

$$f(t) = (-1)^{mN} \prod_{j=1}^{m} \frac{(t - w_j)(t - w_{m+j})}{(t - y_j)(1 - t/z_j)}$$

so that we may apply Day's result with $p = 2m$, $s = q = m$, $\{r_j\} = \{w_j\}$, $\{v_j\} = \{y_j\}$, and $\{u_j\} = \{z_j\}$. A subset $M \subset [1, 2m]$ can be identified with a permutation $\sigma \in \Xi_{m,m}$ via $M = \{\sigma(1), \ldots, \sigma(m)\}$ and $\overline{M} = \{\sigma(m+1), \ldots, \sigma(2m)\}$. Then $r_M = (-1)^m \prod_{k=m+1}^{2m} w_{\sigma(k)}$ and

$$c_M = \frac{\prod_{1 \le \ell, j \le m} (w_{\sigma(j)} - y_\ell)(z_\ell - w_{\sigma(m+j)})}{D(y, z) \prod_{\substack{1 \le j \le m \\ m+1 \le k \le 2m}} (w_{\sigma(j)} - w_{\sigma(k)})}$$

Then, the integral in question is

$$\sum_{\sigma \in \Xi_{m,m}} \prod_{k=m+1}^{2m} w_{\sigma(k)}^N \frac{\prod_{1 \le \ell, j \le m} (w_{\sigma(j)} - y_\ell)(z_\ell - w_{\sigma(m+j)})}{D(y, z) \prod_{\substack{1 \le j \le m \\ m+1 \le k \le 2m}} (w_{\sigma(j)} - w_{\sigma(k)})}$$

Zirnbauer and Nonnemacher have a method for obtaining this formula by representation theory and super symmetry. They use Howe's theory of dual pairs to construct the appropriate super-representation, interpretting (square roots of) characteristic polynomials (determinants) in the denominator as traces and in the numerator as super-traces. The desired formula then follows immediately from Weyl's character formula. The detail

14 Averages of Ratios for symplectic and orthogonal ensembles

In this section we derive a formula for

$$R_{S,N}(A, B) := \int_{USp(2N)} \frac{\prod_{\alpha \in A} \Lambda_X(e^{-\alpha})}{\prod_{\beta \in B} \Lambda_X(e^{-\beta})} \, dX \tag{113}$$

where we assume the $\Re\beta > 0$ and also that $|B| \le N$. Let $A = \{\alpha_1, \ldots, \alpha_L\}$ and $B = \{\beta_1, \ldots, \beta_M\}$. We begin with an algebraic identity from Fyodorov and Strahov. Suppose that $M \le N$. Then

$$M!(N-M)! \prod_{\substack{1\le m\le M\\ 1\le n\le N}} (x_m - y_n)^{-1}$$

$$= \sum_{\sigma\in\pi_N} \frac{\Delta(y_{\sigma(1)},\dots,y_{\sigma(M)})\Delta(y_{\sigma(M+1)},\dots,y_{\sigma(N)}) \prod_{m=1}^M (y_{\sigma(m)}/x_m)^{N-M}}{\Delta(y_{\sigma(1)},\dots,y_{\sigma(N)}) D(\{y_{\sigma(1)},\dots,y_{\sigma(M)}\},\{x_1,\dots,x_M\})}$$

where recall that

$$D(Y,X) = \prod_{\substack{x\in X\\ y\in Y}} (x-y).$$

This identity is a consequence of the Cauchy-Littlewood identity. We apply it with $x_m = \cos b_m$, where we are thinking of $\beta_m = -ib_m$, and $y_n = \cos\theta_n$, using

$$\Lambda_X(e^{ib}) = (2e^{ib})^N \prod_{n=1}^N (\cos b - \cos\theta_n).$$

When we integrate over $USp(2N)$ the answer is independent of the permutation σ. So, we may do the calculation for $\sigma = \mathrm{Id}$ and multiply the result by $N!$. Thus, $R_{S,N}(A,B) =$

$$\frac{2^{N^2+MN} e^{N\sum ib_m}}{\pi^N M!(N-M)!} \int_{[0,\pi]^N} \frac{\prod_{m=1}^M (\cos\theta_m/\cos b_m)^{N-M} \Delta(\cos\theta_1,\dots,\cos\theta_M)}{D(\{\cos\theta_1,\dots,\cos\theta_M\},\{\cos b_1,\dots,\cos b_M\})}$$
$$\times \frac{\Delta(\cos\theta_{M+1},\dots,\cos\theta_N)\Delta(\cos\theta_1,\dots,\cos\theta_N)^2}{\Delta(\cos\theta_1,\dots,\cos\theta_N)} \prod_{\alpha\in A} \Lambda_X(e^{-\alpha}) \prod_{n=1}^N \sin^2\theta_n d\theta_n$$

Now the idea is to integrate over the variables $\theta_{M+1},\dots\theta_N$ and make use of the formula for averages of characteristic polynomials from the last section. If X is the matrix with eigenvalues $\{e^{i\theta_n} : 1 \le n \le N\}$, let X_1 denote a matrix with eigenvalues $\{e^{i\theta_m} : 1 \le m \le M\}$ and X_2 denote a matrix with eigenvalues $\{e^{i\theta_n} : M+1 \le n \le N\}$. We consider the part of the integrand that depends on these latter θ and seek to evaluate

$$\int_{[0,\pi]^{N-M}} \Delta(\cos\theta_{M+1},\dots,\cos\theta_N)\Delta(\cos\theta_1,\dots,\cos\theta_N)$$
$$\times \prod_{\alpha\in A} \Lambda_X(e^{-\alpha}) \prod_{n=M+1}^N \sin^2\theta_n d\theta_n. \tag{114}$$

Note that

$$\Delta(x_1,\ldots,x_M,y_1,\ldots,y_N)=\Delta(x_1,\ldots,x_M)D(\{x_1,\ldots,x_M\};\{y_1,\ldots,y_N\})\Delta(y_1,\ldots,y_N)$$

and

$$D(\{\cos\theta_1,\ldots,\cos\theta_M\},\{\cos\theta_{M+1},\ldots,\cos\theta_N\})=\prod_{m=1}^{M}(2e^{i\theta_m})^{-N}\Lambda_{X_2}(e^{i\theta_m}).$$

Also

$$\Lambda_X(e^{-\alpha})=\Lambda_{X_1}(e^{-\alpha})\Lambda_{X_1}(e^{-\alpha}).$$

Thus, the integral in (114) is

$$\frac{(N-M)!\pi^{N-M}}{2^{(N-M)^2}}\prod_{m=1}^{M}(2e^{i\theta_m})^{-N}\prod_{\alpha\in A}\Lambda_{X_1}(e^{-\alpha}) \times\int_{USp(2N-2M)}\prod_{\ell=1}^{L}\Lambda_{X_2}(e^{-\alpha_\ell})\prod_{m=1}^{M}\Lambda_{X_2}(e^{i\theta_m})dX_2 \qquad (115)$$

Now

$$\int_{USp(2N-2M)}\prod_{\ell=1}^{L}\Lambda_{X_2}(e^{-\alpha_\ell})\prod_{m=1}^{M}\Lambda_{X_2}(e^{i\theta_m})dX_2=M_{S,N-M}(C)$$

where

$$C=\{\alpha_1,\ldots,\alpha_L,-i\theta_1,\ldots,-i\theta_M\}:=\{-i\gamma_1,\ldots,-i\gamma_{L+M}\}$$

where

$$\gamma_\ell=\begin{cases}\theta_\ell & \text{for } 1\le\ell\le L\\ a_{\ell-L} & \text{for } L+1\le\ell\le L+M\end{cases}$$

By (104) we have

$$M_{S,N-M}(C) \;=\; 2^{-M(2M+1)-2}e^{iN\sum_{\ell=1}^{L+M}\gamma_\ell}\frac{\det_{L+M}(U_{N-L+j-1}(\cos\gamma_k))}{\Delta(\cos\gamma_1,\ldots\cos\gamma_{2L})}.$$

Just to be clear, the determinant is of the matrix

$$\begin{pmatrix} U_{N-L}(\cos\theta_1) & \dots & U_{N-L}(\cos\theta_M) & U_{N-L}(\cos a_1) & \dots & U_{N-L}(\cos a_L) \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ U_{N+M-1}(\cos\theta_1) & \dots & U_{N+M-1}(\cos\theta_M) & U_{N+M-1}(\cos a_1) & \dots & U_{N+M-1}(\cos a_L) \end{pmatrix}$$

Inserting this into the expression for R we have

$$R_{S,N}(A,B) \quad = \quad \int_{USp(2M)} \frac{M_{S,N-M}(C)\prod_{m=1}^{M} e^{-iN\theta_m}(\frac{\cos\theta_m}{\cos b_m})^{N-M}\prod_{\ell=1}^{L}\Lambda_{X_1}(e^{-\alpha_\ell})}{D(\{\cos\theta_1,\dots,\cos\theta_M\},\{\cos b_1,\dots,\cos b_M\})} dX_1$$

Now $\prod_{\ell=1}^{L}\Lambda_{X_1}(e^{ia_\ell}) = \prod_{\ell=1}^{L}(2e^{ia_\ell})^M D(\{\cos a_1,\dots,\cos a_L\},\{\cos\theta_1,\dots,\cos\theta_M\})$, and

$$\begin{aligned}\Delta(\cos\gamma_1,\dots\cos\gamma_{L+M}) \quad &= \quad \Delta(\cos a_1,\dots,\cos a_L)\Delta(\cos\theta_1,\dots,\cos\theta_M) \\ &\qquad \times D(\{\cos a_1,\dots,\cos a_M\},\{\cos\theta_1,\dots,\cos\theta_L\})\end{aligned}$$

so that

$$\frac{\prod_{\ell=1}^{L}\Lambda_{X_1}(e^{-\alpha_\ell})}{\Delta(\cos\gamma_1,\dots\cos\gamma_{L+M})} = \frac{2^{MN}e^{M\sum_{\ell=1}^{L} ia_\ell}}{\Delta(\cos a_1,\dots,\cos a_L)\Delta(\cos\theta_1,\dots,\cos\theta_M)}.$$

Thus,

$$R_{S,N}(A,B) \quad = \quad \frac{2^{MN-M(2M+1)-2+M^2}\prod_{m=1}^{M}(\cos b_m)^{M-N}}{\pi^M M!\Delta(\cos a_1,\dots,\cos a_L)} I(B,C) \quad (116)$$

where

$$\begin{aligned} I(B,C) = \int_{[0,\pi]^M} \det_{L+M} U_{N-L+j-1}(\cos\gamma_k)\Delta(\cos\theta_1,\dots,\cos\theta_M) \\ \times\prod_{m=1}^{M} g(\cos\theta_m)(\cos\theta_m)^{N-M}\sin^2\theta_m \; d\theta_m\end{aligned}$$

with

$$g(\cos\theta) = \prod_{n=1}^{M}(\cos b_n - \cos\theta)^{-1}.$$

We can absorb the factor $g(\cos\theta_m)$ into the mth column of the determinantal expression for $\Delta(\cos\theta_1,\dots,\cos\theta_M)$ so that

$$\begin{pmatrix} 1 & \dots & 1 \\ \cos\theta_1 & \dots & \cos\theta_M \\ \vdots & \dots & \vdots \\ \cos^{M-1}\theta_1 & \dots & \cos^{M-1}\theta_M \end{pmatrix} \prod_{m=1}^{M} g(\cos\theta_m)$$

can be rewritten as

$$\det_{M\times M}\left(g(\cos\theta_k)\cos^{j-1}\theta_k\right) \tag{117}$$

We use the partial fraction decomposition

$$\frac{x^r}{\prod_{n=1}^{M}(x-y_n)} = \sum_{n=1}^{M} \frac{y_n^r}{\prod_{k\neq n}(y_\ell - y_k)} \frac{1}{x-y_n}$$

to obtain

$$g(\cos\theta)\cos^{j-1}\theta = \sum_{n=1}^{M} \frac{(\cos b_n)^{j-1}}{(\cos b_n - \cos\theta)} \prod_{\ell\neq n}(\cos b_n - \cos b_\ell)^{-1}.$$

Thus, (117) becomes

$$\begin{aligned} &\frac{1}{\Delta(\cos b_1,\dots,\cos b_M)^2} \det_{M\times M}\left(\sum_{n=1}^{M} \frac{(\cos b_n)^{j-1}}{(\cos b_n - \cos\theta_k)}\right) \\ &= \frac{1}{\Delta(\cos b_1,\dots,\cos b_M)^2} \det_{M\times M}\left((\cos b_k)^{j-1}\right) \det_{M\times M}\left(\frac{1}{(\cos b_j - \cos\theta_k)}\right) \\ &= \frac{1}{\Delta(\cos b_1,\dots,\cos b_M)} \det_{M\times M}\left(\frac{1}{(\cos b_j - \cos\theta_k)}\right) \end{aligned}$$

using that the determinant of a product of two matrices is the product of their determinants. We now have

$$\begin{aligned} I(B,C) &= \int_{[0,\pi]^M} \det_{L+M} U_{N-L+j-1}(\cos\gamma_k) \frac{\det_{M\times M}\left(\frac{1}{(\cos b_j - \cos\theta_k)}\right)}{\Delta(\cos b_1,\dots,\cos b_M)} \\ &\qquad \times \prod_{m=1}^{M} (\cos\theta_m)^{N-M} \sin^2\theta_m \, d\theta_m \end{aligned}$$

This situation allows for the application of the generalized form of Andréief's identity from which we obtain

$$I(B,C) = \frac{1}{\Delta(\cos b_1,\ldots,\cos b_M)} \det_{L+M\times L+M}\left(\mathcal{T}_j \frac{U_{N-M+j-1}(\cos\theta)}{(\cos\theta - \cos b_k)}\right) \tag{118}$$

where

$$\mathcal{T}_j\phi_j(\theta)\psi_k(\theta) = \begin{cases} \int_0^\pi \phi_j(\theta)\psi_k(\theta)\cos^{N-M}\theta\sin^2\theta\ d\theta & \text{if } j \le M \\ \psi_k(a_{j-M}) & \text{if } M < j \le L+M \end{cases}$$

with $\phi_j(\theta) = U_{N-M+j-1}(\cos\theta)$ and $\psi_k(\theta) = (\cos\theta - \cos b_k)^{-1}$. We claim that if $|e^{i\phi}| < 1$ and $R \le k$ then

$$\int_0^\pi \frac{\sin(k+1)\theta\ \sin\theta\ \cos^R\theta}{(\cos\phi - \cos\theta)}\ d\theta = \pi e^{i(k+1)\phi}\cos^R\phi.$$

Proof of Claim. First of all, note that $|e^{i\phi}| < 1$ implies that

$$\sum_{\ell=-\infty}^{\infty} e^{i|\ell|\phi + i\ell\theta} = \frac{-i\sin\phi}{\cos\phi - \cos\theta}.$$

Next, we have

$$2\sin(k+1)\theta\sin\theta = \cos k\theta - \cos(k+2)\theta$$

Now,

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{\cos k\theta\cos^R\theta}{(\cos\phi - \cos\theta)}\ d\theta &= \frac{i2^{-R-1}}{\sin\phi}\sum_{\ell=-\infty}^{\infty}\sum_{r=0}^{R}\binom{R}{r} \\ &\qquad \times \int_{-\pi}^{\pi} e^{i|\ell|\phi + i\ell\theta} e^{ir\theta} e^{-(R-r)\theta}\left(e^{ik\theta} + e^{-ik\theta}\right)\ d\theta \\ &= \frac{2\pi i 2^{-R}}{\sin\phi}\sum_{\ell=-\infty}^{\infty}\sum_{r=0}^{R}\binom{R}{r} e^{i(2r-R+k)\phi} \\ &= \frac{2\pi i(\cos\phi)^R e^{ik\phi}}{\sin\phi} \end{aligned}$$

since $|2r - R| \le k$. Finally

$$\frac{i(e^{ik\phi} - e^{i(k+2)\phi}}{\sin\phi} = e^{i(k+1)\phi}$$

and that completes the proof of the claim. □

We insert this result into (118) to obtain

$$I(B,C) = \frac{\pi^M \prod_{m=1}^M (\cos b_m)^{N-M}}{\Delta(\cos b_1, \dots, \cos b_M)}$$
$$\times \det \begin{pmatrix} 1 & e^{ib_1} & \dots & e^{i(L+M-1)b_1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & e^{ib_M} & \dots & e^{i(L+M-1)b_M} \\ \sinh(N-M+1)\alpha_1 & \sinh(N-M+2)\alpha_1 & \dots & \sinh(N+L)\alpha_1 \\ \vdots & \vdots & \dots & \vdots \\ \sinh(N-M+1)\alpha_L & \sinh(N-M+2)\alpha_L & \dots & \sinh(N+L)\alpha_L \end{pmatrix}$$

Now insert this expression into (116) to obtain

$$R_{S,N}(A,B) = \frac{2^{MN-M(2M+1)-2+M^2}}{M!\Delta(\cos a_1, \dots, \cos a_L)} \frac{}{\Delta(\cos b_1, \dots, \cos b_M)}$$
$$\times \det \begin{pmatrix} 1 & e^{ib_1} & \dots & e^{i(L+M-1)b_1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & e^{ib_M} & \dots & e^{i(L+M-1)b_M} \\ \sinh(N-M+1)\alpha_1 & \sinh(N-M+2)\alpha_1 & \dots & \sinh(N+L)\alpha_1 \\ \vdots & \vdots & \dots & \vdots \\ \sinh(N-M+1)\alpha_L & \sinh(N-M+2)\alpha_L & \dots & \sinh(N+L)\alpha_L \end{pmatrix}$$

These considerations lead to

Theorem 2

$$R_{S,N}(A,B) = \sum_{\substack{1 \le m \le M \\ \epsilon_m \in \{-1,+1\}}} e^{N \sum_{m=1}^M (\epsilon_m \alpha_m - \alpha_m)}$$
$$\times \frac{\prod_{1 \le j \le k \le M} z(\epsilon_j \alpha_j + \epsilon_m \alpha_m) \prod_{1 \le j < k \le |B|} z(\beta_j + \beta_k)}{\prod_{\substack{m \le M \\ \ell \le |B|}} z(\epsilon_m \alpha_m + \beta_\ell)}.$$

15 Symmetric function theory

15.1 Schur polynomials

Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$. Define the Schur polynomial associated with λ by

$$s_\lambda(x) = s_{\lambda_1,\ldots,\lambda_N}(x_1, \ldots, x_N) := \frac{\det(x_j^{k-1+\lambda_k})}{\det(x_j^{k-1})} \tag{119}$$

Then s_λ is a polynomial with integer coefficients, since each factor $x_k - x_j$ of the denominator is also a factor of the numerator. The s_λ form an orthonormal set of functions on $U(N)$. In other words,

$$\int_{U(N)} s_\lambda(e^{i\theta_1}, \ldots, e^{i\theta_N})\overline{s_\mu(e^{i\theta_1}, \ldots, e^{i\theta_N})}dU_N = \delta_{\lambda\mu} \tag{120}$$

where $\delta_{\lambda\mu}$ is 1 if $\lambda = \mu$ and is 0 otherwise.

Proof. We can rewrite the first integral as

$$\int_{[0,2\pi]^N} \det(e^{i\theta_j(k-1+\lambda_k)}) \det(e^{-i\theta_j(k-1+\mu_k)}) \frac{d\theta_1 \ldots d\theta_N}{(2\pi)^N N!}$$

By Andréief's identity, this is

$$\det_{N\times N}\left(\int_0^{2\pi} e^{i\theta(j-1+\lambda_j)} e^{-i\theta(k-1+\mu_k)} \frac{d\theta}{2\pi}\right)$$

Now suppose that $\lambda_1 < \mu_1$. Then the entire first row is 0, since $\lambda_1 < k-1+\mu_k$ for each of $k = 1, 2, \ldots, N$. Therefore the determinant is 0. A similar argument but with the first column shows that $\mu_1 < \lambda_1$ also implies that the determinant is 0. Let's assume now that $\lambda_1 = \mu_1$. Then the 1,1 entry of the determinant is 2π but the rest of the entries in the first row and in the first column of the determinant are all 0. Thus, the determinant reduces to the $N-1 \times N-1$ determinant

$$\det_{N-1\times N-1}\left(\int_0^{2\pi} e^{i\theta(j-1+\lambda'_j)} e^{-i\theta(k-1+\mu'_k)} \frac{d\theta}{2\pi}\right)$$

where $\lambda'_j = \lambda_{j+1} - 1$ and $\mu'_j = \mu_{j+1} - 1$ for $1 \leq j \leq N-1$. Now if $\lambda'_1 \neq \mu'_1$ then the integral is 0; if $\lambda'_1 = \mu'_1$ then we may reduce to an $N-2 \times N-2$ determinant. In this way, we find that the integral is 0 unless $\lambda_j = \mu_j$ for all j, in which case the integral is 1.

Note that since $s_{0,\dots,0} = 1$, it follows that

$$\int_{U(N)} s_\lambda(e^{i\theta_1}, \dots, e^{i\theta_N}) dU_N = \begin{cases} 0 \text{ if } \lambda_N > 0 \\ 1 \text{ if } \lambda_N = 0 \end{cases}$$

15.2 Secular Coefficients

We would like to now describe work of Diaconis and Shahshahani and of Diaconis and Gamburd about moments of the coefficients of characteristic polynomials of unitary matrices. Let us write

$$\Lambda_U(x) = \det(U - Ix) = (-1)^N \sum_{j=0}^{N} \mathrm{Sc}_j(U) x^{N-j} (-1)^j.$$

The coefficients Sc, for some reason, are called the *secular* coefficients. Note that $\mathrm{Sc}_1(U) = \mathrm{Tr}(U)$ and $\mathrm{Sc}_N(U) = \det(U)$. Diaconis and Shahshahani proved the very elegant result that

$$\int_{U(N)} \prod_{j=1}^{\ell} (\mathrm{Tr}(U^j))^{a_j} \overline{(\mathrm{Tr}(U^j))^{b_j}} dU_N = \delta_{a,b} \prod_{j=1}^{\ell} j^{a_j} j!$$

provided only that $N \geq \max(\sum_{j=1}^{\ell} j a_j, \sum_{j=1}^{\ell} j b_j)$. Here $\delta_{a,a} = 1$ and $\delta_{a,b} = 0$ if the sequence $a = (a_1, \dots, a_\ell)$ is not identical to the sequence $b = (b_1, \dots, b_\ell)$. The method of proof is to express the traces in terms of the Schur polynomials and use the orthogonality of these to deduce the result. An interesting generalization was given by Diaconis and Gamburd who considered

$$C_{a,b} = \int_{U(N)} \prod_{j=1}^{\ell} (\mathrm{Sc}_j(U))^{a_j} \overline{(\mathrm{Sc}_j(U))^{b_j}} dU_N$$

for non-negative integers a_j and b_j. Provided again that $N \geq \max(\sum_{j=1}^{\ell} j a_j, \sum_{j=1}^{\ell} j b_j)$, they prove that $C_{a,b}$ is equal to the number of matrices of size $(a_1 + 2a_2 +$

$\cdots + \ell a_\ell) \times (b_1 + 2b_2 + \cdots + \ell b_\ell)$ with non-negative entries for which the first a_1 rows add up to 1, the next a_2 rows add up to 2, and so on until the last a_ℓ rows add up to ℓ; similarly the first b_1 columns must add up to 1, and so on up until the last b_ℓ columns add up to b_ℓ. Thus, the number $C_{a,b}$ counts the number of "magic" rectangles with nonnegative entries and specified row and column sums.

The secular coefficients are naturally expressible in terms of the elementary symmetric functions of the roots (i.e. the eigenvalues). The results described above are proven by first expressing the elementary symmetric functions in terms of the Schur polynomials and then using the orthogonality of the Schur polynomials. To describe this process further it is necessary to introduce some basic background information about symmetric functions.

15.3 Symmetric Polynomials

The collection of homogeneous symmetric polynomials of degree d in k variables $x_1, \ldots, x_k$ form a vector space whose dimension is equal to the number of partitions of d into at most k parts. Given such a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ we adopt the convention that the parts λ_i form a decreasing sequence: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$. Since we allow 0s at the end of our partition, we can assume that our partitions have length exactly k. We let

$$|\lambda| = \lambda_1 + \cdots + \lambda_k. \tag{121}$$

So, when we say that λ is a partition of d we mean that $|\lambda| = d$. Given λ we can form the elementary symmetric function associated with λ which is just

$$e_\lambda = \prod_{j=1}^{k} e_{\lambda_j} \tag{122}$$

where for a number n, we let e_n denote the n elementary symmetric function (in the k variables)

$$e_n = \sum_{1 \leq j_1 < \cdots < j_n \leq k} x_{j_1} \ldots x_{j_n}. \tag{123}$$

If we just have k variables at our disposal, then e_n will be 0 for $n > k$. If we want to emphasize the variables, we write $e_n(x_1, \ldots, x_k)$. We will see that

$\{e_\lambda : |\lambda| = d\}$ forms a basis for the vector space of homogeneous degree d polynomials in k variables.

Another basis for this vector space is given by the complete symmetric polynomials h_λ. These are defined by

$$h_\lambda = \prod_{j=1}^{k} h_{\lambda_j} \tag{124}$$

where for a number n we let

$$h_n = \sum_{1 \le j_1 \le \cdots \le j_n \le k} x_{j_1} \ldots x_{j_n}. \tag{125}$$

Thus, the j_m are allowed to be equal to each other. For comparison, we note the generating functions

$$E(t) := \sum_{n=0}^{k} e_n(x_1, \ldots, x_k) t^n = \prod_{j=1}^{k} (1 + x_j t) \tag{126}$$

and

$$H(t) := \sum_{n=0}^{\infty} h_n(x_1, \ldots, x_k) t^n = \prod_{j=1}^{k} (1 - x_j t)^{-1}. \tag{127}$$

In particular, $H(t)E(-t) = 1$, from which we deduce the identity

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0$$

for $n \ge 1$. This identity is compactly stated as

$$\det_{n \times n}(h_{k-j}) = \det_{n \times n}((-1)^{k-j} e_{k-j}). \tag{128}$$

Another useful collection of symmetric polynomials are the power polynomials $p_\lambda = \prod_{j=1}^{k} p_{\lambda_j}$ where for a number r,

$$p_r = p_r(x_1, \ldots, x_k) = \sum_{j=1}^{k} x_j^r. \tag{129}$$

The generating function is

$$P(t) = \sum_{r=1}^{\infty} p_r t^{r-1} = \sum_{j=1}^{k} \frac{x_j}{1 - x_j t} = -\sum_{j=1}^{k} \frac{d}{dt} \log(1 - x_j t) = \frac{H'(t)}{H(t)} \tag{130}$$

It follows that

$$P(-t) = \frac{E'(t)}{E(t)}$$

also holds. Equating coefficients of these power series we obtain

$$nh_n = \sum_{r=1}^{n} p_r h_{n-r} \tag{131}$$

and

$$ne_n = \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r}; \tag{132}$$

the latter are known as the Newton identities.

There is a more general determinantal identity between h_λ and $e_{\lambda'}$ where λ' is the partition conjugate to λ. (If the partition λ is represented as a collection of left-justified rows of unit squares with λ_1 squares in the first row, λ_2 squares in the second row, and so on (this is called the Young diagram), then the Young diagram for λ' is obtained by transposing the Young diagram for λ so that the columns of λ become the rows of λ'.) The general identity is

$$\det_{n\times n}(h_{\lambda_j+k-j}) = \det_{m\times m}(e_{\lambda'_j+k-j}) = s_\lambda$$

where s_λ is the Schur polynomial; this identity holds as long as m and n are at least as large as the lengths of λ' and λ, respectively. We have seen a special case (with all of the variables equal to 1 and $\lambda = (N, \ldots, N, 0, \ldots 0)$) of this identity before in the determinantal formulae for $g(k, N)$.

Recall that we defined the Schur polynomials as quotients of determinants:

$$s_\lambda(x_1, \ldots, x_k) = \frac{\det_{k\times k}(x_j^{\lambda_i+k-i})}{\det_{k\times k}(x_j^{k-i})}.$$

The Schur polynomials are the characters of the irreducible representations of $U(k)$. These are related to the characters of the symmetric group

π_k of permutations of $\{1, \ldots, k\}$. The characters of π_k are parametrized by partitions λ of k. Let χ_λ denote the character of π_k associated with λ. The fundamental identity gives the expression of the power polynomials in terms of the Schur polynomials with coefficients given by the characters of the symmetric group. Any such character is constant on conjugacy classes of π_k and conjugacy of π_k classes are parametrized by partitions of k in that the cycle decomposition of a permutation naturally gives a partition. Specifically, for any partition μ of k we have

$$p_\mu = \sum_\lambda \chi_\lambda(\mu) s_\lambda$$

where $\chi_\lambda(\mu)$ denotes the value of the character on the conjugacy class determined by μ.

From this formula and the orthonormality of the Schur polynomials, we calculate that

$$\int_{U(N)} p_\mu \overline{p_{\mu'}} dU_N = \sum_{\lambda,\lambda'} \chi_\lambda(\mu)\overline{\chi_{\lambda'}(\mu')} \int_{U(N)} s_\lambda \overline{s_{\lambda'}} dU_N = \sum_\lambda \chi_\lambda(\mu)\overline{\chi_\lambda(\mu')} = \delta_{\mu,\mu'} z_\mu$$

where, by the orthogonality relations for characters, $z_\mu = \prod j^{\alpha_j} j!$ if the partition μ has α_1 ones, α_2 twos and so on.

15.4 Szego Limit Theorem

Szego proved an asymptotic formula for the Toeplitz determinant of order n whose entries are generated by the Laurent coefficients of certain functions. A version of what is called the Strong Szego theorem (see Bump and Diaconis) is as follows. Let $\sum_{k=-\infty}^{\infty} |c_k| < \infty$ and $\sum_{k=-\infty}^{\infty} |k||c_k|^2 < \infty$. Then

$$\lim_{n\to\infty} e^{-nc_0} \det_{n\times n}(d_{k-j}) = e^{\sum_{k=1}^{\infty} kc_k c_{-k}} \tag{133}$$

where

$$\exp\Big(\sum_{k=-\infty}^{\infty} c_k t^k\Big) = \sum_{n=-\infty}^{\infty} d_k t^k.$$

Actually there is an identity discovered by several authors which implies this:

$$e^{-nc_0} \det_{n\times n}(d_{k-j}) = e^{\sum_{k=1}^{\infty} kc_k c_{-k}} \det(I - K_n) \tag{134}$$

where

$$K_n = (I - P_n)H(\phi_-/\phi_+)H(\tilde{\phi_+/\phi_-})(I - P_n)$$

16 Distribution of values of characteristic polynomials

16.1 Distribution of small values

Let $V_N(x)$ be the probability that a characteristic polynomial in $U(N)$, evaluated on the unit circle, has absolute value smaller than x, i.e.

$$V_N(x) = \text{meas}\{U \in U(N) : |\Lambda_U(1)| \leq x\}.$$

Our complete knowledge of the moments of $|\Lambda_U(1)|$ allows us to give an exact formula for $V_N(x)$.

We use the classical Perron formula in the form

$$\frac{1}{2\pi i}\int_{(c)} \frac{-x^s}{s}\, ds = \begin{cases} 1 & \text{if } x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Here $c < 0$ (say $c = -1/8$) and (c) denotes the straight line path from $c - i\infty$ to $c + i\infty$ Let

$$M_N(s) = \int_{U(N)} |\Lambda_U(1)|^s\, ds = g(\frac{s}{2}, N).$$

Then

$$\begin{aligned} V_N(x) &= \frac{-1}{2\pi i}\int_{(c)} M_N(s)x^{-s}\frac{ds}{s} \\ &= \frac{-1}{2\pi i}\int_{(c)} \prod_{j=1}^{N} \frac{\Gamma(j+s)\Gamma(j)}{\Gamma(j+\frac{s}{2})^2} x^{-s}\frac{ds}{s} \end{aligned} \tag{135}$$

Now by Stirling's formula, $\Gamma(j+s)/\Gamma(j+s/2)^2 \ll |s|^{\frac{1}{2}-j}$, so the integral is convergent.

The integrand has a simple pole at $s=-1$ a triple pole at $s=-3$, a fifth order pole at $s=-5$ and so on. If $x<1$ we can move the path leftward, crossing these poles and express $V(x)$ as a convergent series

$$V(x) = c_1(N)x + x^3(c_{3,2}(N)\log^2 x + c_{3,1}(N)\log x + c_{3,0}(N)) + \dots$$

where

$$c_0(N) = \frac{1}{\pi}\prod_{j=2}^{N}\frac{\Gamma(j-1)\Gamma(j)}{\Gamma(j-1/2)} \sim c\log^{1/4} N.$$

16.2 Normal distribution of values

Keating and Snaith obtained precise information about the distribution of $\log|\Lambda_{U,N}(1)|$. Its normal distribution can easily be deduced from the formula for the moments of this characteristic polynomial. It can also be deduced from a theorem of Basor [B] about the asymptotics of Toeplitz determinants with a Fisher-Hartwig symbol. This result may be stated as

$$\lim_{N\to\infty} \text{meas } \{U \in U(N) : \log|\Lambda_U(1)| \le \lambda\sqrt{\log N}\} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\lambda} e^{-t^2}\, dt. \quad (136)$$

Keating and Snaith obtain much more precise information. For complete details, see the paper [KS] or the Snaith's thesis [Sna].

To begin with, we record an asymptotic formula (see [Hu1]) for

$$M_{U,N}(s) = g\big(\frac{s}{2}, N\big) = \int_{U(N)} |\Lambda_U(1)|^s \, dU_N = \frac{G(\frac{s}{2}+1)^2 G(N+1)G(N+1+s)}{G(s+1)G^2(\frac{s}{2}+1+N)} \quad (137)$$

Recall (90): For large $|z|$ with $|\arg(z)|<\pi$

$$\log G(z+1) = Az^2\log z + Bz^2 + Cz + D\log z + E + O(1/|z|)$$

where $A=1/2$, $B=-3/4$, $C=\frac{1}{2}\log 2\pi$, $D=-\frac{1}{12}$, and $E=\zeta'(-1)$. Also, $\log M_{U,N}(s)$

$$= 2\log G(1+\tfrac{s}{2}) + \log G(1+N) + \log G(1+N+s) - \log G(1+s) - 2\log G(1+N+\tfrac{s}{2}).$$

We now compute an asymptotic formula for $\log M_N(s)$. Let $Q(z) = Bz^2 + Cz$ and $f(N) = 2Q(s/2) + Q(N) + Q(N+s) - Q(s) - 2Q(N+s/2)$. Then f is a quadratic function of N and $f(0) = f'(0) = f''(0) = 0$. Therefore, $f(N) \equiv 0$ and so the B and C terms vanish. The contribution of the D-terms is

$$D \log \frac{\frac{s^2}{4} N(N+s)}{s(N+\frac{s}{2})^2} = D \log \frac{sN(N+s)}{(s+2N)^2}, \tag{138}$$

and the contribution of the E-terms is just E. Finally, the contribution of the A-terms is $A\times$

$$\begin{aligned} &2\frac{s^2}{4} \log \frac{s}{2} + N^2 \log N + (N+s)^2 \log(N+s) - s^2 \log s - 2(N+\tfrac{s}{2})^2 \log(N+\tfrac{s}{2}) \\ &= \frac{s^2}{2} \log \frac{(s+N)^2}{s(s+2N)} + N^2 \log \frac{N(N+s)}{(N+\frac{s}{2})^2} + 2sN \log \frac{N+s}{N+\frac{s}{2}}. \end{aligned}$$

Thus, for $-1 < \Re s < 1$,

$$\begin{aligned} \log M_{U,N}(s) \;=\; & \frac{s^2}{4} \log \frac{(s+N)^2}{s(s+2N)} + \tfrac{1}{2} N^2 \log \frac{N(N+s)}{(N+\frac{s}{2})^2} + sN \log \frac{N+s}{N+\frac{s}{2}} \\ & - \frac{1}{12} \log \frac{Ns(N+s)}{(s+2N)^2} + \zeta'(-1) + O\left(\frac{1}{|s|} + \frac{1}{N}\right) \end{aligned} \tag{139}$$

We describe the behavior of $\log M_N(s)$ for various ranges of s. For very large s, i.e. $N = o(|s|)$, we have

$$\log M_N(s) \sim \frac{s^2}{4}. \tag{140}$$

For large N, i.e. $|s| = o(N)$, we have

$$\log M_N(s) \sim \frac{s^2}{4} \log \frac{N}{2s}. \tag{141}$$

For s and N comparable size, say $s = it$ with $N = \lambda t$, we have

$$\Re \log M_N(it) = -\frac{t^2}{4} f(\lambda) - \frac{1}{12} \log \frac{\lambda\sqrt{1+\lambda^2}}{1+4\lambda^2} + \zeta'(-1) + O((N^2+t^2)^{-1/2}) \tag{142}$$

where

$$f(\lambda) = \frac{1}{4} \log \frac{1+\lambda^2}{\sqrt{1+4\lambda^2}} - \frac{\lambda^2}{2} \log \frac{\lambda\sqrt{1+\lambda^2}}{\lambda^2+\frac{1}{4}} + \lambda \arctan \frac{1}{\lambda} - \lambda \arctan \frac{1}{2\lambda} \tag{143}$$

Note that $f(\lambda) > 0$ for $\lambda > 0$; also, $f(\lambda)$ is monotonically increasing and $f(\lambda) \sim \lambda^2 \log \frac{1}{\lambda}$ as $\lambda \to 0^+$. Finally, though we don't use it here, we note the more precise estimate for large N,

$$\log M_N(s) = \frac{s^2}{4} \log \frac{N}{2s} + \frac{3}{8} s^2 - \frac{1}{12} \log \frac{s}{4} + \zeta'(-1) + O\left(\frac{(1+|s|^3)}{N} + \frac{1}{|s|} \right) \tag{144}$$

Recall (135) that

$$V_N(x) = \frac{-1}{2\pi i} \int_{(c)} M_N(s) x^{-s} \frac{ds}{s} = \int_0^x \left(\frac{1}{2\pi i} \int_{(0)} M_N(s) t^{-s} \, ds \right) \frac{dt}{t} \tag{145}$$

It follows from the estimate (140) that the integral over s is very rapidly convergent (because $\Re(s^2) = (\Re s)^2 - (\Im s)^2$). By a change of variables

$$V_N\left(e^{\lambda \sqrt{\log N}}\right) = \int_{-\infty}^{\lambda} \frac{\sqrt{\log N}}{2\pi i} \int_{(0)} M_N(s) e^{-su\sqrt{\log N}} \, ds \, du. \tag{146}$$

Thus, to prove (136), it suffices to prove that

$$\lim_{N\to\infty} \frac{\sqrt{\log N}}{2\pi i} \int_{(0)} M_N(s) e^{-su\sqrt{\log N}} \, ds = \frac{e^{-u^2}}{\sqrt{\pi}}. \tag{147}$$

Now express $\log M_N(s) = \frac{s^2}{4} \log N + g_N(s)$. Since $M_N(0) = 1$, it follows that $g_N(0) = 0$. Note that

$$\frac{s^2}{4} \log N - su\sqrt{\log N} = -u^2 + \frac{\log N}{4} \left(s - \frac{2u}{\sqrt{\log N}} \right)^2 \tag{148}$$

so that the left side of (147) is

$$\begin{aligned} &\lim_{N\to\infty} \frac{\sqrt{\log N} e^{-u^2}}{2\pi i} \int_{(0)} e^{\frac{\log N}{4}\left(s - \frac{2u}{\sqrt{\log N}}\right)^2} e^{g_N(s)} \, ds \\ &= \lim_{N\to\infty} \frac{\sqrt{\log N} e^{-u^2}}{2\pi i} \int_{(0)} e^{\frac{\log N}{4} s^2} e^{g_N(s + 2u/\sqrt{\log N})} \, ds \end{aligned} \tag{149}$$

by a change of variable and Cauchy's theorem. Now let $w = \frac{\sqrt{\log N}}{2} s$. Then (149) becomes

$$\lim_{N\to\infty} \frac{e^{-u^2}}{\pi i} \int_{(0)} e^{w^2} e^{g_N(2(u+w)/\sqrt{\log N})} \, dw \tag{150}$$

Using the estimates for $\log M_N(s)$ given above, it is not difficult to check that

$$\Re\big(w^2 + g_N(2(u+w)/\sqrt{\log N})\big) \tag{151}$$

is uniformly bounded in w and N for any fixed u. Therefore, we may take the limit as $N \to \infty$ inside the integral and obtain that (149) becomes

$$\frac{e^{-u^2}}{\pi i} \int_{(0)} e^{w^2} \, dw = \frac{e^{-u^2}}{\sqrt{\pi}}. \tag{152}$$

17 Distribution of zeros and values of derivatives of characteristic polynomials

17.1 Motivation

Characteristic polynomials of unitary matrices are extremely useful models for the Riemann zeta-function $\zeta(s)$. In particular, the distribution of their eigenvalues give useful insight into the distribution of zeros of the Riemann zeta-function and the values of these characteristic polynomials give a model for the value distribution of $\zeta(s)$.

A problem of interest in number theory is to determine the horizontal distribution of zeros of $\zeta'(s)$. Knowledge of this distribution is the key element in Levinson's famous proof that more than 1/3 of the zeros of $\zeta(s)$ have real part equal to 1/2.

To elaborate, we recall that the Riemann Hypothesis is the assertion that all non-real zeros of $\zeta(s)$ have real part 1/2. Speiser proved that the Riemann Hypothesis is equivalent to the assertion that all non-real zeros of $\zeta'(s)$ have real part greater than or equal to 1/2. It is not difficult to show that if $\zeta'(1/2+i\gamma) = 0$ for a real number γ then $\zeta(1/2+i\gamma) = 0$; in words, the derivative of zeta vanishes on the 1/2-line only at a multiple zero of zeta. It is widely believed that all of the zeros of $\zeta(s)$ are simple. Consequently, it is expected that all of the non-real zeros of $\zeta'(s)$ will lie strictly to the right of the 1/2-line.

The point of departure for Levinson's celebrated work was a theorem of Levinson and Montgomery [LM] asserting that up to a height T above the real axis, $zeta(s)$ and $\zeta'(s)$ have the same number of zeros strictly to the left of the 1/2-line, apart from a small number $O(\log T)$ possible exceptions. Consequently, if the proportion of zeros of $\zeta'(s)$ to the left of the 1/2-line is at most δ, then the proportion of zeros of $\zeta(s)$ to the left of the 1/2-line is also at most δ. The zeros of $\zeta(s)$ are symmetric about the 1/2-line. Hence, the proportion of zeros of $\zeta(s)$ to the right of the 1/2-line is also at most δ. Then the proportion of zeros of $\zeta(s)$ on the 1/2-line must be at least $1-2\delta$. Levinson set out to find an upper bound for δ.

Levinson proved the inequality

$$\frac{1}{N(T)}\sum_{\substack{\gamma'\leq T\\ \beta'<a}}(a-\beta')\leq\frac{0.429}{\log T}$$

where $\beta'+i\gamma'$ is a generic zero of $\zeta'(s)$ and $N(T)$ is the number of zeros of $\zeta(s)$ (and essentially $\zeta'(s)$) up to a height T and $a=1/2+\frac{1.3}{\log T}$. The left side of this inequality is clearly an upper bound for $\delta(a-1/2)=\frac{1.3\delta}{\log T}$. Consequently, $\delta\leq 0.33$ and so Levinson deduced that at least 34% of the zeros of $\zeta(s)$ are on the critical-line.

It is fairly clear from studying the approach that Levinson's inequality is not sharp. One would like to know precisely the value of

$$\sum_{\substack{\gamma'\leq T\\ \beta'<a}}(a-\beta')$$

for any value of a that is of the form $1/2+\frac{\alpha}{\log T}$. A related question is, what proportion $f(\alpha)$ of the zeros of $\zeta'(s)$ with ordinates up to T have real parts between 1/2 and $a=1/2+\alpha/\log T$?

The purpose of this chapter is to begin the study of zeros of the derivatives of the characteristic polynomials of unitary matrices that are supposed to model $\zeta(s)$. We find that this question is not so easy to answer in this setting either. For $N\times N$ unitary matrices, all of the zeros of the characteristic polynomials are on the unit circle and all of the zeros of the derivative are inside or on the unit circle. The question in this setting asks for the radial distribution of these zeros measuring on a scale of $1/N$ from the unit circle. We expect that $f(\alpha)$ will be the proportion of the zeros of the derivative in the annulus with inner radius $1-\alpha/N$ and outer radius 1. Francesco

Mezzadri has the best results in this direction; see [Mez] where he obtains asymptotic formulas for $f(\alpha)$ as $\alpha \to 0$ and also obtains estimates for large α.

On the number theory side, partial results have been obtained by Levinson and Montgomery [LM], Conrey and Ghosh [CG], Soundararajan [Sou], and Zhang [Z].

Our approach to this problem is to find precise information about (all of the complex) moments of the derivatives of characteristic polynomials. Then, from this information use Jensen's formula to deduce the required formulae. Chris Hughes [Hug] has made some progress toward this goal by considering even integral moments of the derivative of a characteristic polynomial; he has found explicit formula for all integer k. We adopt a method different from Hughes and end up with different sets of formulae. However, neither our results nor Hughes' seem able to provide the desired analytic continuation that would allow us to infer non-integral real or complex moments. The hope is that the description of our results may stimulate others to consider the problem.

Thus, the results in this chapter are somewhat fragmentary and represent a modest beginning toward what could be a really interesting theory.

17.2 Discrete Moments

We outline an argument of Chris Hughes which evaluates moments of derivatives of characteristic polynomials evaluated at their zeros. Let $Z(\theta) = \Lambda(e^{-i\theta}) = \prod_{j=1}^{N}(1 - e^{i(\theta_j - \theta)})$. Consider

$$J_s = \int_{U(N)} \sum_{j=1}^{N} |Z'(\theta_j)|^s.$$

Then, for $\Re s > -3$,

$$J_s = \frac{G^2(\frac{s}{2}+2)G(N+s+2)G(N)}{G(s+3)G^2(N+\frac{s}{2}+1)}.$$

Proof. By an easy calculation,

$$|Z'(\theta_j)| = \prod_{\substack{n=1 \\ n\neq j}}^{N} |e^{i\theta_j} - e^{i\theta_n}|.$$

By symmetry, the above formula, and the definition of Haar measure,

$$J_s = N\int_{U(N)} |Z'(\theta_N)|^s = \frac{1}{(N-1)!(2\pi)^N}\int_{[0,2\pi]^N} |\Delta(e^{i\theta_1},\ldots,e^{i\theta_{N-1}})|^2 \prod_{j=1}^{N-1} |e^{i\theta_j} - e^{i\theta_N}|^{s+2} d\theta_j.$$

Now consider the integral over $\theta_1,\ldots,\theta_{N-1}$. It is just the $s+2$ moment of $|Z(\theta_N)|$, which by rotational invariance of the Haar measure on the unitary group is the same as the $s+2$ moment of $|Z(0)| = |\Lambda(1)|$. Thus,

$$J_s = \frac{1}{2\pi}\int_{[0,2\pi]} g(s+2, N-1) = \frac{1}{N} g(s+2, N-1) = \frac{G^2(\frac{s}{2}+2)G(N+s+2)G(N)}{G(s+3)G^2(N+\frac{s}{2}+1)}$$

as desired.

Now consider (as Hughes did) the more general "shifted discrete moment"

$$J(\alpha,\beta) = \int_{U(N)} \sum_{h=1}^{N}\prod_{j=1}^{m} \Lambda_U(e^{-i\theta_h-\alpha_j}) \prod_{k=1}^{n} \Lambda_{U^*}(e^{-i\theta_h-\beta_k}) dU_N.$$

Just as above, we reduce this to a moment problem over $U(N-1)$:

$$\begin{aligned} J(\alpha,\beta) &= N\int_{U(N)} \prod_{j=1}^{m} \Lambda_U(e^{-i\theta_N-\alpha_j}) \prod_{k=1}^{n} \Lambda_{U^*}(e^{-i\theta_N-\beta_k}) dU_N \\ &= \frac{1}{2\pi}\int_{[0,2\pi]}\int_{U(N-1)} \prod_{j=1}^{m} \Lambda_B(e^{-i\theta_N-\alpha_j}) \prod_{k=1}^{n} \Lambda_{B^*}(e^{-i\theta_N-\beta_k}) \Lambda_B(e^{-i\theta_N})\Lambda_{B^*}(e^{i\theta_N}) dB\ d\theta_N \end{aligned}$$

where we have temporarily adopted the notation that B is obtained from U by fixing the Nth eigenvalue to be θ_N and we have used the fact that

$$\begin{aligned} \prod_{h=1}^{N-1} |e^{i\theta_N} - e^{i\theta_h}|^2 &= \prod_{h=1}^{N-1} |1 - e^{i\theta_h - \theta_N}|^2 \\ &= \prod_{h=1}^{N-1} (1 - e^{i\theta_h-\theta_N})(1 - e^{-i\theta_h+\theta_N}) = \Lambda_B(e^{-i\theta_N})\Lambda_{B^*}(e^{i\theta_N}). \end{aligned}$$

By rotational invariance of the Haar measure, the integral over $U(n-1)$ above gives the same value for each θ_N. (Just replace each θ_h by $\theta_h + \theta_N$ for $1 \le h \le N-1$ and the measure dU_{N-1} is unaffected.) Thus, the above simplifies to

$$J(\alpha,\beta) = \int_{U(N-1)} \prod_{j=1}^{m} \Lambda_B(e^{-\alpha_j}) \prod_{k=1}^{n} \Lambda_{B^*}(e^{-\beta_k}) \Lambda_B(1)\Lambda_{B^*}(1) dB.$$

Now we want to apply our results on shifted moments to the integral over $U(N-1)$. In our earlier notation, we have

$$J(\alpha,\beta) = I_{N-1}(\alpha \cup 0, \beta \cup 0)$$

where we have inserted the subscript $N-1$ on I to denote the dimension of the integral.

One way to apply our earlier results about $I_N(\alpha,\beta)$ requires that we perturb our question and consider

$$J_\epsilon(\alpha,\beta) = I_{N-1}(\alpha \cup \epsilon, \beta \cup \epsilon);$$

then we will use $\lim_{\epsilon\to 0} J_\epsilon = J$. Then we can apply our earlier results about I and simplify the result by using the identity $z(a) + z(-a) = 1$; the way this identity is used is in the form

$$\begin{aligned} \lim_{\epsilon\to 0}(G(\epsilon)z(\epsilon) + e^{-(N-1)2\epsilon}G(-\epsilon)z(-\epsilon)) &= \lim_{\epsilon\to 0}((G(\epsilon) - e^{-(N-1)2\epsilon}G(-\epsilon))z(\epsilon) + G(-\epsilon)) \\ &= G'(0)/2 + NG(0) \end{aligned}$$

for any function G which is differentiable at 0; this fact follows from $z(\epsilon) = 1/\epsilon + O(1)$.

Alternatively, we can proceed directly to an integral formula and obtain:

$$J(\alpha,\beta) = \frac{(-1)^{m+n} e^{\sum_{j,k}\alpha_j+\beta_k}}{(m+1)!(n+1)!} \int_{|w_i|=c} \frac{e^{-N\sum_{h=1}^{m+n+2} w_h} \Delta(w)^2 \prod_{\substack{j\le m+1\\ k\le n+1}} z(w_h + w_{m+1+k})}{\prod_{h\le m+n}(w_h^2 \prod_{j=1}^{m}(w_h-\alpha_j)\prod_{k=1}^{n}(w_h-\beta_k))} \prod dw_i$$

where $c = 2\max|\alpha_j|, |\beta_k|$.

17.3 Moments of derivatives of characteristic polynomials

We let

$$z(x) = \frac{1}{1-e^{-x}} = \frac{1}{x} + O(1). \tag{153}$$

The function $z(x)$ plays the role for random matrix theory that $\zeta(1+x)$ plays in the theory of moments of the Riemann zeta-function.

We let I_n be the usual modified Bessel function with power series expansion

$$I_n(x) = \left(\frac{x}{2}\right)^n \sum_{j=0}^{\infty} \frac{x^{2j}}{2^{2j}(n+j)!j!}. \tag{154}$$

The way that the I-Bessel function enters our calculation is through the following formula:

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{e^{Lz+t/z}}{z^{2k}}\, dz = \frac{L^{2k-1} I_{2k-1}(2\sqrt{Lt})}{(Lt)^{k-1/2}}; \tag{155}$$

this formula can be proven by comparing the coefficient of z^{2k-1} in $e^{Lz+t/z}$ with the power series formula for I_{2k-1}.

We let Ξ denote the subset of permutations $\sigma \in S_{2k}$ of $\{1, 2, \ldots, 2k\}$ for which

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k) \tag{156}$$

and

$$\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(2k). \tag{157}$$

We let $P_O^{k+1}(2k)$ be the number of partitions $m = (m_0, \ldots, m_k)$ of $2k$ into $k+1$ non-negative parts. This quantity arises from the multinomial expansion

$$(x_0 + x_1 + \cdots + x_k)^{2k} = \sum_{m \in P_O^{k+1}(2k)} \binom{2k}{m} x_0^{m_0} \cdots x_k^{m_k} \tag{158}$$

where

$$\binom{2k}{m} = \frac{(2k)!}{m_0! \dots m_k!}. \tag{159}$$

In its simplest form our problem is to give an exact formula, valid for complex s with $\Re s > 0$, of

$$K_{N,s} := \int_{U(N)} |\Lambda'_U(1)|^s dU_N. \tag{160}$$

or of

$$K'_{N,s} := \int_{U(N)} |\mathcal{Z}'_U(1)|^s dU_N. \tag{161}$$

Unfortunately, we cannot yet solve either of these problems. However, we can give explicit formulae for $K_{N,2k}$ and $K'_{N,2k}$ for positive integer values of k.

Theorem 3 *For fixed k and $N \to \infty$ we have*

$$K_{N,2k} \sim (-1)^{k(k+1)/2} b_k N^{k^2+2k} \tag{162}$$

where

$$b_k = \sum_{h=0}^{k} \binom{k}{h} \left(\frac{d}{dx}\right)^{k+h} \left(e^{-x} x^{-k^2/2} \det_{k\times k} \left(I_{i+j-1}(2\sqrt{x}) \right) \right) \Bigg|_{x=0} \tag{163}$$

Theorem 4 *For fixed k and $N \to \infty$ we have*

$$K'_{N,2k} \sim (-1)^{k(k+1)/2} b'_k N^{k^2+2k} \tag{164}$$

where

$$b'_k = \left(\frac{d}{dx}\right)^{2k} \left(e^{-\frac{x}{2}} x^{-k^2/2} \det_{k\times k} \left(I_{i+j-1}(2\sqrt{x}) \right) \right) \Bigg|_{x=0} \tag{165}$$

We also have combinatorial description of b'_k.

Theorem 5

$$b'_k = \sum_{m \in P_O^{k+1}(2k)} \binom{2k}{m} \left(\frac{-1}{2}\right)^{m_0} \left(\prod_{i=1}^{k} \frac{1}{(2k-i+m_i)!}\right) \left(\prod_{1 \le i < j \le k} (m_j - m_i + i - j)\right) \quad (166)$$

We have computed some values of b_k and b'_k; these are tabulated at the end of the paper.

Lemma 7 (CFKRS1) *We have*

$$\int_{U(N)} \prod_{j=1}^{k} \Lambda_U(e^{-\alpha_j}) \Lambda_{U^*}(e^{\alpha_{j+k}})\ dU_N = \sum_{\sigma \in \Xi} e^{N \sum_{j=1}^{k} (\alpha_{\sigma(j)} - \alpha_j)} \prod_{1 \le i,j \le k} z(\alpha_{\sigma(j)} - \alpha_{\sigma(k+i)}) \quad (167)$$

Since

$$\mathcal{Z}_U(e^{-\alpha_j}) \mathcal{Z}_{U^*}(e^{\alpha_{j+k}}) = (-1)^N e^{N(\alpha_j - \alpha_{j+k})/2} \Lambda_U(e^{-\alpha_j}) \Lambda_{U^*}(e^{\alpha_{j+k}}) \quad (168)$$

we have

Lemma 8

$$\int_{U(N)} \prod_{j=1}^{k} \mathcal{Z}_U(e^{-\alpha_j}) \mathcal{Z}_{U^*}(e^{\alpha_{j+k}})\ dU_N \quad (169)$$

$$= e^{-\frac{N}{2} \sum_{j=1}^{k} \alpha_j} \sum_{\sigma \in \Xi} e^{N \sum_{j=1}^{k} \alpha_{\sigma(j)}} \prod_{1 \le i,j \le k} z(\alpha_{\sigma(j)} - \alpha_{\sigma(k+i)}) \quad (170)$$

We can express the sums in the last two lemmas as integrals. Thus we have

Lemma 9 *Assume that all of the α_j are smaller than 1 in absolute value. Then*

$$\int_{U(N)} \prod_{j=1}^{k} \Lambda_U(e^{-\alpha_j}) \Lambda_{U^*}(e^{\alpha_{j+k}})\ dU_N \quad (171)$$

$$= \frac{1}{k!(2\pi i)^k} \int_{|w_i|=1} e^{N \sum_{j=1}^{k} (w_j - \alpha_j)} \prod_{\substack{1 \le i \le k \\ 1 \le j \le 2k}} z(w_i - \alpha_j) \prod_{i \ne j} z(w_i - w_j)^{-1} \prod_{j=1}^{k} dw_j \quad (172)$$

and

$$\int_{U(N)} \prod_{j=1}^{k} \mathcal{Z}_U(e^{-\alpha_j})\mathcal{Z}_{U^*}(e^{\alpha_{j+k}})\ dU_N \tag{173}$$

$$= \frac{e^{-\frac{N}{2}\sum_{j=1}^{2k}\alpha_j}}{k!(2\pi i)^k} \int_{|w_i|=1} e^{N\sum_{j=1}^{k} w_j} \prod_{\substack{1\le i\le k\\ 1\le j\le 2k}} z(w_i-\alpha_j) \prod_{i\neq j} z(w_i-w_j)^{-1} \prod_{j=1}^{k} dw_j \tag{174}$$

Using the fact that $z(w) = 1/w + O(1)$ we easily deduce

Corollary 2 *Suppose that $|\alpha_j| \ll 1/N$ for each j. Then*

$$\int_{U(N)} \prod_{j=1}^{k} \Lambda_U(e^{-\alpha_j})\Lambda_{U^*}(e^{\alpha_{j+k}})\ dU_N \tag{175}$$

$$= \frac{1}{k!(2\pi i)^k} \int_{|w_i|=1} e^{N\sum_{j=1}^{k}(w_j-\alpha_j)} \frac{\prod_{i\neq j}(w_i-w_j)}{\prod_{\substack{1\le i\le k\\ 1\le j\le 2k}}(w_i-\alpha_j)} \prod_{j=1}^{k} dw_j + O(N^{k^2-1}) \tag{176}$$

with an implicit constant independent of N; similarly,

$$\int_{U(N)} \prod_{j=1}^{k} \mathcal{Z}_U(e^{-\alpha_j})\mathcal{Z}_{U^*}(e^{\alpha_{j+k}})\ dU_N \tag{177}$$

$$= \frac{e^{-\frac{N}{2}\sum_{j=1}^{2k}\alpha_j}}{k!(2\pi i)^k} \int_{|w_i|=1} e^{N\sum_{j=1}^{k} w_j} \frac{\prod_{i\neq j}(w_i-w_j)}{\prod_{\substack{1\le i\le k\\ 1\le j\le 2k}}(w_i-\alpha_j)} \prod_{j=1}^{k} dw_j + O(N^{k^2-1}) \tag{178}$$

Lemma 10 *Let f be an infinitely differentiable function. Then*

$$\Delta^2\Big(\frac{d}{dL}\Big)\Big(\prod_{j=1}^{k} f(L_j)\Big)\Big|_{L_j=L} = k! \det_{k\times k}\big(f^{(i+j-2)}(L)\big) \tag{179}$$

where by $\Delta(d/dL)$ we mean the differential operator

$$\prod_{1\le h<j\le k}\Big(\frac{d}{dL_j}-\frac{d}{dL_h}\Big) = \det_{k\times k}\Big(\big(\frac{d}{dL_i}\big)^{j-1}\Big). \tag{180}$$

More generally, suppose that $g(L_1,\dots,L_k)=\sum_{r=1}^R a_r\prod_{i=1}^k f_{r,i}(L_i)$ *is a symmetric function of its* k *variables. Then*

$$\Delta^2\Big(\frac{d}{dL}\Big)g(L_1,\dots,L_K)\Big|_{L_j=L} = k!\sum_{r=1}^R a_r \det_{k\times k}\big(f_{r,i}^{(i+j-2)}(L)\big) \tag{181}$$

Lemma 11 *Suppose that* P *and* Q *are polynomials with* $Q(w)=\prod_{j=1}^{2k}(w-\alpha_j)$ *and* $\max|\alpha_j|<c$. *Then*

$$\frac{1}{2\pi i}\int_{|w|=c}\frac{e^{wL}}{w}\frac{P(w)}{Q(w)}ds = P\Big(\frac{d}{dL}\Big)\int_{\sum_{j=1}^{2k}x_j\le L} e^{\sum_{j=1}^{2k}x_j\alpha_j}\prod_{j=1}^{2k}dx_j. \tag{182}$$

Lemma 12 *We have*

$$\int_{\sum_{j=1}^{2k}x_j\le L} x_1\dots x_n\prod_{j=1}^{2k}dx_j = \frac{L^{2k+n}}{(2k+n)!}. \tag{183}$$

We now give the proofs of our identities for the leading terms of the moments of the derivatives of Λ and $\mathcal{Z}$. We begin with the proof of Theorem 2 for $\mathcal{Z}$ as it is slightly easier.

Proof of Theorem 2. A differentiated form of the second formula of the Corollary gives us

$$\prod_{j=1}^{2k}\frac{d}{d\alpha_j}\int_{U(N)}\prod_{h=1}^k \mathcal{Z}_U(e^{-\alpha_h})\mathcal{Z}_{U^*}(e^{\alpha_k+h})dU_N = (-1)^{\frac{k(k+1)}{2}}\mathcal{K}'_{N,2k}(\alpha)+O(N^{k^2+k-1}),$$

provided that $\alpha_j \ll 1/N$, where

$$\mathcal{K}'_{N,2k}(\alpha)=\prod_{j=1}^{2k}\frac{d}{d\alpha_j}\frac{e^{-\frac{N}{2}\sum_{j=1}^{2k}\alpha_j}}{k!(2\pi i)^k}\int_{|w_i|=1}e^{N\sum_{j=1}^k w_j}\frac{\Delta^2(w)}{\prod_{\substack{1\le i\le k\\1\le j\le 2k}}(w_i-\alpha_j)}\prod_{j=1}^k dw_j. \tag{184}$$

Now we introduce variables L_i and, using our earlier notation $\mathcal{K}'_{N,2k}$, find that

$$\mathcal{K}'_{N,2k}=\prod_{j=1}^{2k}\frac{d}{d\alpha_j}\frac{\Delta^2(d/dL)e^{-\frac{N}{2}\sum_{j=1}^{2k}\alpha_j}}{k!(2\pi i)^k}\int_{|w_j|=1}\frac{e^{\sum_{i=1}^k L_iw_i}}{\prod_{\substack{1\le i\le k\\1\le j\le 2k}}(w_i-\alpha_j)}\prod_{j=1}^k dw_j\Big|_{\alpha_j=0,L_i=N}$$

Next, we observe that

$$\frac{d}{d\alpha}\frac{e^{-\frac{N}{2}\alpha}}{\prod_{1\le i\le k}(w_i-\alpha)}\bigg|_{\alpha=0}=\frac{1}{\prod_{i=1}^k w_i}\left(\sum_{j=1}^k\frac{1}{w_j}-\frac{N}{2}\right) \tag{185}$$

so that

$$\mathcal{K}'_{N,2k}=\frac{\Delta^2(d/dL)}{k!(2\pi i)^k}\int_{|w_j|=1}\frac{e^{\sum_{i=1}^k L_iw_i}\left(\sum_{j=1}^k\frac{1}{w_j}-\frac{N}{2}\right)^{2k}}{\prod_{i=1}^k w_i^{2k}}\prod_{j=1}^k dw_j\bigg|_{L_i=N} \tag{186}$$

Introducing another auxiliary variable t, this can be expressed as

$$\begin{aligned}\mathcal{K}'_{N,2k} &= \frac{\Delta^2(d/dL)\left(\frac{d}{dt}\right)^{2k}e^{-Nt/2}}{k!(2\pi i)^k}\int_{|w_j|=1}\frac{e^{\sum_{i=1}^k L_iw_i+t/w_i}}{\prod_{i=1}^k w_i^{2k}}\prod_{j=1}^k dw_j\bigg|_{L_i=N,t=0} &(187)\\ &= \frac{\Delta^2(d/dL)\big(d/dt\big)^{2k}e^{-Nt/2}}{k!}\prod_{i=1}^k\left(\frac{1}{2\pi i}\int_{|w|=1}\frac{e^{L_iw+t/w}}{w^{2k}}dw\right)\bigg|_{L_i=N,t=0} &(188)\end{aligned}$$

The integral evaluates to

$$\frac{L_i^{2k-1}I_{2k-1}(2\sqrt{L_it})}{(L_it)^{k-1/2}} \tag{189}$$

as noted earlier. Thus,

$$\mathcal{K}'_{N,2k} = \frac{\Delta(d/dL)\big(d/dt\big)^{2k}e^{-Nt/2}}{k!}\left(\prod_{i=1}^k\frac{L_i^{2k-1}I_{2k-1}(2\sqrt{L_it})}{(L_it)^{k-1/2}}\right)\bigg|_{L_i=N,t=0} \tag{190}$$

So, letting

$$f_t(L)=\frac{L_i^{2k-1}I_{2k-1}(2\sqrt{L_it})}{(L_it)^{k-1/2}}, \tag{191}$$

we have, by Lemma 4,

$$\mathcal{K}'_{N,2k}=\left(\frac{d}{dt}\right)^{2k}e^{-Nt/2}\left(\det_{k\times k}\left(f_t^{(i+j-2)}(N)\right)\right)\bigg|_{t=0}. \tag{192}$$

Now we calculate that

$$f_t(L) = \sum_{r=0}^{\infty} \frac{t^r L^{2k-1+r}}{r!(2k-1+r)!} \tag{193}$$

so that if $\mu \le 2k-1$, then

$$f^{(\mu)}(L) = \sum_{r=0}^{\infty} \frac{t^r L^{2k-1-\mu+r}}{r!(2k-1-\mu+r)!} = \left(\frac{L}{t}\right)^{(2k-1-\mu)/2} I_{2k-1-\mu}(2\sqrt{Lt}). \tag{194}$$

Thus,

$$\mathcal{K}'_{N,2k} = \left(\frac{d}{dt}\right)^{2k} e^{-Nt/2} \det_{k\times k}\left(\left(\frac{N}{t}\right)^{(2k+1-i-j)/2} I_{2k+1-i-j}(2\sqrt{Nt})\right)\Bigg|_{t=0}. \tag{195}$$

Clearly $\det_k(a_{i,j}) = \det_k(a_{k+1-i,k+1-j})$. Thus, we have

$$\mathcal{K}'_{N,2k} = \left(\frac{d}{dt}\right)^{2k} e^{-Nt/2} \det_{k\times k}\left(\left(\frac{N}{t}\right)^{(i+j-1)/2} I_{i+j-1}(2\sqrt{Nt})\right)\Bigg|_{t=0}. \tag{196}$$

If we substitute $x = Nt$, then $d/dt = Nd/dx$ and we have

$$\begin{aligned}\mathcal{K}'_{N,2k} &= N^{2k}\left(\frac{d}{dx}\right)^{2k} e^{-x/2} \det_{k\times k}\left(\left(\frac{N^2}{x}\right)^{(i+j-1)/2} I_{i+j-1}(2\sqrt{x})\right)\Bigg|_{x=0} &(197)\\ &= N^{k^2+2k}\left(\frac{d}{dx}\right)^{2k}\left(e^{-x/2}x^{-k^2/2} \det_{k\times k}\left(I_{i+j-1}(2\sqrt{x})\right)\right)\Bigg|_{x=0} &(198)\end{aligned}$$

since $\det_k(M^{i+j-1}a_{i,j}) = M^{k^2}\det_k(a_{i,j})$ as is seen by factoring M^j out of the jth column and M^{i-1} out of the ith row. This completes the proof of Theorem 2.

Proof of Theorem 1. Turning to Theorem 1's proof, we begin as before, but with a differentiated form of the first formula of the Corollary:

$$\prod_{j=1}^{k} \frac{d}{d\alpha_j} \int_{U(N)} \prod_{h=1}^{k} \Lambda_U(e^{-\alpha_h})\Lambda_{U^*}(e^{\alpha_{k+h}})dU_N = (-1)^{\frac{k(k+1)}{2}} \mathcal{K}_{N,2k}(\alpha) + O(N^{k^2+k-1}), \tag{199}$$

provided that $\alpha_j \ll 1/N$, where

$$\mathcal{K}_{N,2k}(\alpha) = \prod_{j=1}^{k} \frac{d}{d\alpha_j} \frac{1}{k!(2\pi i)^k} \int_{|w_i|=1} e^{N\sum_{j=1}^{k}(w_j-\alpha_j)} \frac{\Delta^2(w)}{\prod_{\substack{1\le i\le k\\ 1\le j\le 2k}}(w_i-\alpha_j)} \prod_{j=1}^{k} dw_j, \tag{200}$$

Introducing variables L_i as before, we have

$$\mathcal{K}_{N,2k} = \frac{\prod_{j=1}^{2k}(d/d\alpha_j)\Delta^2(d/dL)}{k!(2\pi i)^k}\int_{|w_j|=1}\frac{e^{\sum_{i=1}^k(L_iw_i-N\alpha_i)}}{\prod_{\substack{1\le i\le k\\1\le j\le 2k}}(w_i-\alpha_j)}\prod_{j=1}^k dw_j\Big|_{\alpha_j=0,L_i=N} \quad (201)$$

Performing the differentiations with respect to the α_j leads us to

$$\mathcal{K}_{N,2k} = \frac{\Delta^2(d/dL)}{k!(2\pi i)^k}\int_{|w_j|=1}\frac{e^{\sum_{i=1}^k L_iw_i}\left(\sum_{j=1}^k\frac{1}{w_j}-N\right)^k\left(\sum_{j=1}^k\frac{1}{w_j}\right)^k}{\prod_{i=1}^k w_i^{2k}}\prod_{j=1}^k dw_j\Big|_{L_i=N} \quad (202)$$

Now we write

$$\left(\sum_{j=1}^k\frac{1}{w_j}-N\right)^k\left(\sum_{j=1}^k\frac{1}{w_j}\right)^k = \left(\sum_{j=1}^k\frac{1}{w_j}-N\right)^k\left(\sum_{j=1}^k\frac{1}{w_j}-N+N\right)^k \quad (203)$$

$$= \sum_{h=0}^k\binom{k}{h}N^{k-h}\left(\sum_{j=1}^k\frac{1}{w_j}-N\right)^{k+h}. \quad (204)$$

Introducing the auxiliary variable t, this can be expressed as

$$\mathcal{K}_{N,2k} = \sum_{h=0}^k\binom{k}{h}N^{k-h}\frac{\Delta^2(d/dL)\left(\frac{d}{dt}\right)^{k+h}e^{-Nt}}{k!(2\pi i)^k}\int_{|w_j|=1}\frac{e^{\sum_{i=1}^k L_iw_i+t/w_i}}{\prod_{i=1}^k w_i^{2k}}\prod_{j=1}^k dw_j\Big|_{L_i=N,t=0}$$

$$= \frac{\Delta^2(d/dL)\left(d/dt\right)^{2k}e^{-Nt}}{k!}\prod_{i=1}^k\left(\frac{1}{2\pi i}\int_{|w|=1}\frac{e^{L_iw+t/w}}{w^{2k}}dw\right)\Big|_{L_i=N,t=0}.$$

Proceeding as before we arrive at

$$\mathcal{K}_{N,2k} = N^{k^2+2k}\sum_{h=0}^k\binom{k}{h}\left(\frac{d}{dx}\right)^{k+h}\left(e^{-x}x^{-k^2/2}\det_{k\times k}\left(I_{i+j-1}(2\sqrt{x})\right)\right)\Big|_{x=0} \quad (205)$$

Proof of Theorem 3. We now give the proof of Theorem 3. We rewrite equation (184) as

$$\mathcal{K}'_{N,2k}(\alpha) = \prod_{j=1}^{2k}\frac{d}{d\alpha_j}\frac{e^{-\frac{N}{2}\sum_{j=1}^{2k}\alpha_j}}{k!(2\pi i)^k}\int_{|w_i|=1}e^{N\sum_{i=1}^k w_i}\frac{\Delta^2(w)\prod_{i=1}^k w_i}{\prod_{\substack{1\le i\le k\\1\le j\le 2k}}(w_i-\alpha_j)}\prod_{i=1}^k\frac{dw_i}{w_i} \quad (206)$$

Introducing variables L_i as before, we can rewrite this as

$$\frac{1}{k!}\prod_{j=1}^{2k}\frac{d}{d\alpha_j}e^{-\frac{N}{2}\sum_{j=1}^{2k}\alpha_j}\Delta^2\left(\frac{d}{dL}\right)\prod_{i=1}^{k}\left(\frac{d}{dL_i}\right)\prod_{i=1}^{k}\left(\frac{1}{2\pi i}\int_{|w|=1}\frac{e^{L_i w}}{\prod_{j=1}^{2k}(w-\alpha_j)}\frac{dw}{w}\right) \tag{207}$$

Now, by Lemma 5, the integral is

$$\int_{\sum_{j=1}^{2k} x_j \le L_i} e^{\sum_{j=1}^{2k} x_j\alpha_j}\prod_{1\le j\le 2k} dx_j.$$

Letting the variables in the ith integral be $x_{i,j}$ we may express the product of the k integrals as

$$\int_{\sum_{j=1}^{2k} x_{1,j}\le L_1}\cdots\int_{\sum_{j=1}^{2k} x_{k,j}\le L_k} e^{\sum_{i=1}^{k}\sum_{j=1}^{2k} x_{i,j}\alpha_j}\prod_{\substack{1\le i\le k\\ 1\le j\le 2k}} dx_{i,j}.$$

We incorporate the factor $e^{-\frac{N}{2}\sum_{j=1}^{2k}\alpha_j}$ into this product and have

$$\int_{\sum_{j=1}^{2k} x_{1,j}\le L_1}\cdots\int_{\sum_{j=1}^{2k} x_{k,j}\le L_k} e^{\sum_{j=1}^{2k}\alpha_j\left(\sum_{i=1}^{k} x_{i,j}-N/2\right)}\prod_{\substack{1\le i\le k\\ 1\le j\le 2k}} dx_{i,j}.$$

We differentiate this product of integrals with respect to each α_j and set each α_j equal to 0 yielding

$$\int_{\sum_{j=1}^{2k} x_{1,j}\le L_1}\cdots\int_{\sum_{j=1}^{2k} x_{k,j}\le L_k}\prod_{j=1}^{2k}\left(\sum_{i=1}^{k} x_{i,j}-\frac{N}{2}\right)\prod_{\substack{1\le i\le k\\ 1\le j\le 2k}} dx_{i,j}.$$

We want to compute this integral by multiplying out the product and using Lemma 6. A good way to think about this is as follows. By equation (158)

$$(A_1+\cdots+A_k-A)^{2k}=\sum_{m\in P_O^{k+1}(2k)}\binom{2k}{m}(-A)^{m_0}A_1^{m_1}\cdots A_k^{m_k}.$$

When we multiply out the product we will have a sum of $(k+1)^{2k}$ terms, each term being a product of some number of factors $(-N/2)$ and $x_{i,j}$. Let $m\in P_O^{k+1}(2k)$ represent a generic term in which $(-N/2)$ appears m_0 times, and factors $x_{1,j}$ appear for m_1 values of j, and $x_{2,j}$ for m_2 values of j and so on. When we apply Lemma 6 to this term, when we perform the integration over the variables $x_{1,j}$ the answer is solely determined by m_1, the number of different $x_{1,j}$ that appear in this term. Therefore, we find that the product of integrals evaluates as

$$\sum_{m\in P_O^{k+1}(2k)} \binom{2k}{m} \left(-\frac{N}{2}\right)^{m_0} \frac{L_1^{2k+m_1}}{(2k+m_1)!} \cdots \frac{L_k^{2k+m_k}}{(2k+m_k)!}.$$

We now have that the quantity in equation (207) is equal to

$$\frac{1}{k!}\Delta^2\left(\frac{d}{dL}\right)\prod_{i=1}^{k}\left(\frac{d}{dL_i}\right) \sum_{m\in P_O^{k+1}(2k)} \binom{2k}{m} \left(-\frac{N}{2}\right)^{m_0} \frac{L_1^{2k+m_1}}{(2k+m_1)!} \cdots \frac{L_k^{2k+m_k}}{(2k+m_k)!}.$$

Now we need to carry out the differentiations with respect to the L_i and set the L_i equal to N. We perform the differentiations $\prod_{i=1}^{k} d/dL_i$ and obtain

$$\frac{1}{k!}\Delta^2\left(\frac{d}{dL}\right) \sum_{m\in P_O^{k+1}(2k)} \binom{2k}{m} \left(-\frac{N}{2}\right)^{m_0} \frac{L_1^{2k-1+m_1}}{(2k-1+m_1)!} \cdots \frac{L_k^{2k+m_k}}{(2k+m_k)!}.$$

Now the sum over $m_1, \dots m_k$ is a symmetric function of the variables L_i. Therefore, we can apply the second part of Lemma 4 to obtain that the above, evaluated at $L_i = N$ is

$$\begin{aligned} K'_{N,2k} &= \sum_{m\in P_O^{k+1}(2k)} \binom{2k}{m} \left(-\frac{N}{2}\right)^{m_0} \det_{k\times k}\left(\frac{N^{2k+1+m_i-i-j}}{(2k+1+m_i-i-j)!}\right) \\ &= N^{k^2+2k} \sum_{m\in P_O^{k+1}(2k)} \binom{2k}{m} 2^{-m_0} \det_{k\times k}\left(\frac{1}{(2k+1+m_i-i-j)!}\right) \end{aligned}$$

We factor $1/(2k-i+m_i)!$ out of the ith row. The remaining determinant has ith row

$$1,\ (2k-i+m_i),\ (2k-i+m_i)(2k-i-1+m_i),\quad \dots,\quad \prod_{j=1}^{k-1}(2k-i-j+1+m_i)$$

This determinanat is a polynomial in the m_i of degree $0+1+\cdots+(k-1) = k(k-1)/2$ which vanishes whenever $m_j - m_i = j - i$; moreover the part of it with degree $k(k-1)/2$ is precisely $\Delta(m_1, \ldots, m_k) = \prod_{1 \le i < j \le k}(m_j - m_i)$. Consequently the determinant evaluates to

$$\prod_{1 \le i < j \le k} (m_j - m_i - j + i).$$

This concludes the evaluation of b'_k.

We have the following values for b_k:

$$b_1 = \frac{1}{3}$$

$$b_2 = \frac{61}{2^5 \cdot 3^2 \cdot 5 \cdot 7}$$

$$b_3 = \frac{277}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11}$$

$$b_4 = \frac{2275447}{2^{18} \cdot 3^{10} \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13}$$

$$b_5 = \frac{3700752773}{2^{26} \cdot 3^{14} \cdot 5^6 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19}$$

$$b_6 = \frac{3654712923689}{2^{39} \cdot 3^{19} \cdot 5^9 \cdot 7^6 \cdot 11^3 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23}$$

$$b_7 = \frac{53 \cdot 13008618017 \cdot 143537}{2^{50} \cdot 3^{28} \cdot 5^{13} \cdot 7^8 \cdot 11^5 \cdot 13^4 \cdot 17^2 \cdot 19^2 \cdot 23}$$

$$b_8 = \frac{41 \cdot 359 \cdot 5505609492791 \cdot 3637}{2^{68} \cdot 3^{35} \cdot 5^{16} \cdot 7^{11} \cdot 11^6 \cdot 13^5 \cdot 17^3 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31}$$

$$b_9 = \frac{757 \cdot 45742439 \cdot 60588179 \cdot 13723}{2^{84} \cdot 3^{42} \cdot 5^{21} \cdot 7^{14} \cdot 11^8 \cdot 13^6 \cdot 17^4 \cdot 19^3 \cdot 23^2 \cdot 29 \cdot 31}$$

$$b_{10} = \frac{652071900673 \cdot 241845775551409}{2^{105} \cdot 3^{55} \cdot 5^{25} \cdot 7^{17} \cdot 11^{10} \cdot 13^8 \cdot 17^5 \cdot 19^4 \cdot 23^3 \cdot 29 \cdot 37}$$

$$b_{11} = \frac{1318985497 \cdot 578601141598041214011811}{2^{121} \cdot 3^{64} \cdot 5^{31} \cdot 7^{19} \cdot 11^{12} \cdot 13^9 \cdot 17^7 \cdot 19^6 \cdot 23^4 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43}$$

$$b_{12} = \frac{113 \cdot 2064896333864479201751 41 \cdot 51839 \cdot 14831}{2^{150} \cdot 3^{75} \cdot 5^{37} \cdot 7^{23} \cdot 11^{15} \cdot 13^{12} \cdot 17^{7} \cdot 19^{7} \cdot 23^{5} \cdot 29^{3} \cdot 31^{2} \cdot 37 \cdot 41 \cdot 43 \cdot 47}$$

$$b_{13} = \frac{467075406940462287190406806708963525 4838677}{2^{174} \cdot 3^{90} \cdot 5^{42} \cdot 7^{28} \cdot 11^{17} \cdot 13^{14} \cdot 17^{10} \cdot 19^{9} \cdot 23^{6} \cdot 29^{3} \cdot 31^{3} \cdot 37^{2} \cdot 41 \cdot 43 \cdot 47}$$

$$b_{14} = \frac{107 \cdot 19494604668845559534677934 1 \cdot 996075171809335069}{2^{203} \cdot 3^{103} \cdot 5^{50} \cdot 7^{31} \cdot 11^{20} \cdot 13^{17} \cdot 17^{12} \cdot 19^{10} \cdot 23^{7} \cdot 29^{4} \cdot 31^{4} \cdot 37^{2} \cdot 41 \cdot 43 \cdot 47 \cdot 53}$$

$$b_{15} = \frac{29547975377 \cdot 3981541 \cdot 1807995588661527603489333681461 \cdot 1584311}{2^{230} \cdot 3^{117} \cdot 5^{57} \cdot 7^{37} \cdot 11^{22} \cdot 13^{19} \cdot 17^{14} \cdot 19^{12} \cdot 23^{9} \cdot 29^{5} \cdot 31^{5} \cdot 37^{3} \cdot 41^{2} \cdot 43^{2} \cdot 47 \cdot 53 \cdot 59}$$

We have the following values for b'_k:

$$\begin{aligned}
b'_1 &= \frac{1}{2^2 \cdot 3} \\
b'_2 &= \frac{1}{2^6 \cdot 3 \cdot 5 \cdot 7} \\
b'_3 &= \frac{1}{2^{12} \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11} \\
b'_4 &= \frac{31}{2^{20} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13} \\
b'_5 &= \frac{227}{2^{30} \cdot 3^{12} \cdot 5^6 \cdot 7^4 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19} \\
b'_6 &= \frac{67 \cdot 1999}{2^{42} \cdot 3^{19} \cdot 5^9 \cdot 7^6 \cdot 11^3 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23} \\
b'_7 &= \frac{43 \cdot 46663}{2^{56} \cdot 3^{28} \cdot 5^{13} \cdot 7^8 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23} \\
b'_8 &= \frac{46743947}{2^{72} \cdot 3^{34} \cdot 5^{16} \cdot 7^{11} \cdot 11^6 \cdot 13^4 \cdot 17^3 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31} \\
b'_9 &= \frac{19583 \cdot 16249}{2^{90} \cdot 3^{42} \cdot 5^{21} \cdot 7^{14} \cdot 11^8 \cdot 13^6 \cdot 17^3 \cdot 19^3 \cdot 23^2 \cdot 29 \cdot 31} \\
b'_{10} &= \frac{3156627824489}{2^{110} \cdot 3^{55} \cdot 5^{25} \cdot 7^{17} \cdot 11^{10} \cdot 13^8 \cdot 17^5 \cdot 19^4 \cdot 23^3 \cdot 29 \cdot 31 \cdot 37} \\
b'_{11} &= \frac{59 \cdot 11332613 \cdot 33391}{2^{132} \cdot 3^{63} \cdot 5^{31} \cdot 7^{18} \cdot 11^{12} \cdot 13^{10} \cdot 17^5 \cdot 19^5 \cdot 23^4 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43} \\
b'_{12} &= \frac{241 \cdot 251799899121593}{2^{156} \cdot 3^{75} \cdot 5^{37} \cdot 7^{23} \cdot 11^{15} \cdot 13^{12} \cdot 17^8 \cdot 19^7 \cdot 23^4 \cdot 29^3 \cdot 31^2 \cdot 41 \cdot 43 \cdot 47} \\
b'_{13} &= \frac{285533 \cdot 37408704134429}{2^{182} \cdot 3^{90} \cdot 5^{42} \cdot 7^{28} \cdot 11^{17} \cdot 13^{14} \cdot 17^{10} \cdot 19^8 \cdot 23^5 \cdot 29^3 \cdot 31^3 \cdot 37^2 \cdot 41 \cdot 43 \cdot 47} \\
b'_{14} &= \frac{197 \cdot 1462253323 \cdot 6616773091}{2^{210} \cdot 3^{100} \cdot 5^{50} \cdot 7^{31} \cdot 11^{20} \cdot 13^{17} \cdot 17^{12} \cdot 19^{10} \cdot 23^7 \cdot 29^4 \cdot 31^4 \cdot 37^2 \cdot 41 \cdot 43 \cdot 47 \cdot 53} \\
b'_{15} &= \frac{16255375825174687265195 45837}{2^{240} \cdot 3^{117} \cdot 5^{57} \cdot 7^{37} \cdot 11^{22} \cdot 13^{19} \cdot 17^{14} \cdot 19^{11} \cdot 23^9 \cdot 29^5 \cdot 31^5 \cdot 37^3 \cdot 41^2 \cdot 43^2 \cdot 47 \cdot 53 \cdot 59}
\end{aligned}$$

18 On a first moment

We continue our study of moments. Recall that

$$\Lambda_A(s) = \det(1 - sA^*) = \prod_{n=1}^{N}(1 - se^{-i\theta_n})$$

and the analogue of Hardy's Z-function

$$\mathcal{Z}_A(s) = e^{-i\pi N/2} e^{i\sum_{n=1}^{N}\theta_n/2} s^{-N/2} \Lambda_A(s) \tag{208}$$

which is real on the unit circle and has the same absolute value as $\Lambda_A(s)$ there.

In this section we give Brian Winn's proof of a Conjecture of Conrey and Ghosh which arose from studying the Riemann zeta-function:

Theorem 6

$$\mathcal{I} := \int_{U(N)} |\mathcal{Z}_A(1)\mathcal{Z}_A'(1|\; dA \sim \frac{e^2 - 5}{4\pi} N^2$$

The analogous theorem for the Riemann zeta-function was proven in [CG] under the assumption of the Riemann Hypothesis. The proof uses heavily the expansion of the functions into Dirichlet seris and so it wasn't clear until recently how to prove the analogous result in Random Matrix Theory.

We write

$$\mathcal{I} = \int_{U(N)} |\Lambda_A(1)|^2 \left|\frac{\mathcal{Z}_A'(1)}{\mathcal{Z}_A(1)}\right| \; dA.$$

By (208) we have

$$\begin{aligned} \left|\frac{\mathcal{Z}_A'(1)}{\mathcal{Z}_A(1)}\right| &= \left|-\frac{N}{2} + \frac{\Lambda_A'(1)}{\Lambda_A(1)}\right| \\ &= \left|-\frac{N}{2} + \sum_{n=1}^{N} \frac{e^{-i\theta_n}}{1 - e^{-i\theta_n}}\right| \\ &= \frac{1}{2}\left|\sum_{n=1}^{N} \cot\frac{\theta_n}{2}\right|. \end{aligned}$$

Thus,

$$\mathcal{I} = \frac{1}{2N!(2\pi)^N} \int_{[0,2\pi]^N} \prod_{n=1}^{N} |(1-e^{-i\theta_n})|^2 \left| \prod_{1\le j<k\le N} (e^{i\theta_k} - e^{i\theta_j}) \right|^2 \left| \sum_{n=1}^{N} \cot \frac{\theta_n}{2} \right| d\theta_1 \dots d\theta_N$$

We substitute $x_j = cot\frac{\theta_j}{2}$ (see the section on moments of characteristic polynomials for more details) and have

$$\mathcal{I} = \frac{2^{N^2+2N-1}}{(2\pi)^N N!} \int_{(\infty,\infty)^N} \prod_{n=1}^{N} \frac{1}{(1+x_n^2)^{N+1}} |x_1 + \cdots + x_N| \Delta(x)^2 dx_1 \dots dx_N.$$

We can write this as

$$\lim_{\epsilon\to 0^+} \frac{2^{N^2+2N-1}}{(2\pi)^N N!} \int_{(\infty,\infty)^N} \prod_{n=1}^{N} \frac{1}{(1+x_n^2)^{N+1}} |x_1 + \cdots + x_N| e^{-\epsilon|x_1+\cdots+x_N|} \Delta(x)^2 dx_1 \dots dx_N.$$

The following lemma is easy to prove.

Lemma 13 *Let*

$$K(\epsilon,\zeta) := -\frac{1}{\pi}\frac{\partial}{\partial\epsilon}\left(\frac{\epsilon}{\epsilon^2+\zeta^2}\right).$$

Then

$$\int_{-\infty}^{\infty} K(\epsilon,\zeta) e^{ix\zeta} d\zeta = |x| e^{-\epsilon|x|}.$$

Using this lemma we have

$$\begin{aligned} \mathcal{I} &= \lim_{\epsilon\to 0^+} \frac{2^{N^2+2N-1}}{(2\pi)^N N!} \int_{(-\infty,\infty)^N} \prod_{n=1}^{N} \frac{1}{(1+x_n^2)^{N+1}} |x_1 + \cdots + x_N| e^{-\epsilon|x_1+\cdots+x_N|} \Delta(x)^2 dx_1 \dots dx_N \\ &= \lim_{\epsilon\to 0^+} \frac{2^{N^2+2N-1}}{(2\pi)^N N!} \int_{(-\infty,\infty)^{N+1}} K(\epsilon,\zeta) \prod_{n=1}^{N} \frac{e^{i\zeta x_n}}{(1+x_n^2)^{N+1}} \Delta(x)^2 d\zeta dx_1 \dots dx_N. \end{aligned}$$

We relate this to Laguerre polynomials, which are defined by

$$L_N^{(\alpha)}(t) : = \frac{e^t}{t^\alpha N!}\frac{d^N}{dt^N}\left(t^{\alpha+N}e^{-t}\right) = \sum_{j=0}^{N} \frac{\Gamma(N+\alpha+1)}{\Gamma(j+\alpha+1)(N-j)!}\frac{(-t)^j}{j!}.$$

Lemma 14 *We have*

$$\int_{(-\infty,\infty)^N} \prod_{n=1}^{N} \frac{e^{i\zeta x_n}}{(1+x_n^2)^{N+1}} \Delta(x)^2 dx_1 \dots dx_N = \frac{(2\pi)^N N!}{2^{N^2+2N}} e^{-N|\zeta|} L_N^{(1)}(-2|\zeta|).$$

Thus,

$$\mathcal{I} \;=\; \lim_{\epsilon\to 0^+} \frac{1}{2} \int_{(-\infty,\infty)} K(\epsilon,\zeta) e^{-N|\zeta|} L_N^{(1)}(-2|\zeta|) d\zeta.$$

By the definition of K, this is

$$\mathcal{I} \;=\; -\lim_{\epsilon\to 0^+} \frac{1}{2\pi} \int_{(-\infty,\infty)} \frac{\partial}{\partial \epsilon} \left(\frac{\epsilon}{\epsilon^2+\zeta^2} \right) e^{-N|\zeta|} L_N^{(1)}(-2|\zeta|) d\zeta.$$

Integrating by parts, this is

$$\begin{aligned}
\mathcal{I} \;&=\; -\lim_{\epsilon\to 0^+} \frac{1}{2\pi} \int_{(-\infty,\infty)} \left(\frac{\zeta}{\epsilon^2+\zeta^2} \right) \frac{\partial}{\partial \zeta} e^{-N|\zeta|} L_N^{(1)}(-2|\zeta|) d\zeta \\
&=\; -\lim_{\epsilon\to 0^+} \frac{1}{2\pi} \int_{(-\infty,\infty)} \left(\frac{\zeta}{\epsilon^2+\zeta^2} \right) e^{-N|\zeta|} \left(N L_N^{(1)}(-2|\zeta|) - 2 L_N^{(2)}(-2|\zeta|) \right) d\zeta
\end{aligned}$$

where we have used

$$\frac{d}{dt} L_N^{(\alpha)}(t) = -L_{N-1}^{(\alpha+1)}(t).$$

The recurrence formula is

$$N L_N^{(1)}(-2\zeta) - 2 L_{N-1}^{(2)}(-2\zeta) = 2\zeta L_{N-1}^{(3)}(-2\zeta).$$

This gives

$$\begin{aligned}
\mathcal{I} \;&=\; \lim_{\epsilon\to 0^+} \frac{2}{\pi} \int_0^\infty \frac{\zeta^2}{\epsilon^2+\zeta^2} e^{-N\zeta} L_{N-1}^{(3)}(-2\zeta)\ d\zeta \\
&=\; \frac{2}{\pi} \int_0^\infty e^{-N\zeta} L_{N-1}^{(3)}(-2\zeta)\ d\zeta \\
&=\; \frac{2}{\pi} \sum_{n=0}^{N-1} \binom{N+2}{n+3} \frac{2^n}{N^{n+1}}.
\end{aligned}$$

This series can be summed to give

$$\mathcal{I} = \frac{2}{\pi} \left(\frac{1}{8} \left(N^2 \left(\frac{N+2}{N} \right)^N - 5N^2 + 4N \left(\frac{N+2}{N} \right)^N + 4 \left(\frac{N+2}{N} \right)^N - 10N - 4 \right) \right).$$

From this we see that

$$\lim_{N\to\infty}\frac{\mathcal{I}}{N^2}=\frac{e^2-5}{4\pi}.$$

It remains to prove Lemma 14.

19 Other

- Brian Winn's calculation of $\int_{U(N)}|ZZ'|$
- ratios implies correlations and Rudnick - Sarnak formula
- Diaconis - Shahshahani theorem on traces
- proof of exact Szego theorm and alternate way of doing moments (see Estelle's article form the Newton)
- Keating - Snaith for orthogonal and symplectic; formulas for $g_S(K,N)$, $g_O(K,N)$; integrality
- operator approach to Painleve a la Tracy - Widom
- Yang's approach to Painleve via deteminants
- pictures - nearest neighbor; first eigenvalue for orthogonal etc. (Do we need to do Painleve for lowest eigenvalue?)
- nearest neighbor for $N\times N$ ensemble
- Updated status on Francesco's problem
- Keating and Snaith analogue of moments of S(t) (from their first paper)
- log normal distribution of values of $\Lambda_A(1)$
- does Tracy-Widom tell us anything about extreme gaps between eigenvalues?

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