PAIR CORRELATION OF DIFFERENT L-FUNCTIONS

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1. Some conjectures

We begin by describing a conjecture for an integral of a ratio of two L-functions and then use this conjecture to derive a formula for the correlation between the zeros of the two L-functions.

Let $L_1(s)$ and $L_2(s)$ be two L-functions

$$L_1(s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} \qquad\qquad L_2(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

of degrees d_i with functional equations

$$Q_i^s \prod_{j=1}^{d_i} \Gamma(s/2 + \mu_{i,j}) L_i(s) = \Phi_i(s) = \epsilon_i \overline{\Phi_i}(1-s)$$

for $i = 1, 2$, where $Q_i > 0$. In the assymmetric form we express these functional equations as

$$L_i(s) = \epsilon_i X_i(s) \overline{L_i}(1-s)$$

with

$$X_i(s) = Q_i^{1-2s} \frac{\prod_{j=1}^{d} \Gamma((1-s)/2 + \overline{\mu_{i,j}})}{\prod_{j=1}^{d} \Gamma(s/2 + \mu_{i,j})}$$

Further, we assume that both L-functions are expressible as Euler products

$$L_1(s) = \prod_p \prod_{j=1}^{d} \left(1 - \frac{\alpha_{j,p}}{p^s}\right)^{-1}$$

where $|\alpha_{j,p}| = 1$ unless $p \mid q_1$ in which case either $|\alpha_{j,p}| = 1$ or $\alpha_{j,p} = 0$; here $q_1 = \pi^d Q_1^2$ is assumed to be a positive integer (called the "level" of L_1). Similarly, for L_2 and numbers $\beta_{j,p}$ and the level q_2. Consider

$$R_\zeta(\alpha, \beta, \gamma, \delta) := \int_0^T \frac{L_1(s+\alpha) L_2(1-s+\beta)}{L_1(s+\gamma) L_2(1-s+\delta)} \, dt$$

To derive a conjecture for this integral, we use "approximate functional equations" to expand the numerator L-functions:

$$L_1(s+\alpha) \approx \sum_{m \le x} \frac{a_m}{m^{s+\alpha}} + \epsilon_1 X_1(s+\alpha) \sum_{m \le y} \frac{\overline{a_m}}{m^{1-s-\alpha}}$$

and

$$L_2(1-s+\beta) \approx \sum_{n\le x} \frac{b_n}{n^{1-s+\beta}} + \epsilon_2 X_2(1-s+\beta) \sum_{n\le y} \frac{\overline{b_n}}{n^{s-\beta}}$$

and we expand the denominators into Dirichlet series:

$$\frac{1}{L_1(s+\gamma)} = \sum_{h=1}^{\infty} \frac{a'_h}{h^{s+\gamma}} \qquad \frac{1}{L_2(1-s+\delta)} = \sum_{k=1}^{\infty} \frac{b'_k}{k^{1-s+\delta}}$$

We expand the integral into four pieces using the approximate functional equations. Then, in each piece, we integrate term-by-term and retain only the non-oscillatory (in t-aspect) terms. Here we will restrict our attention to the situation when $d_1 = d_2$ so that we have two L-functions of the same degree. In this case, only two of the four pieces have non-oscillatory terms: The piece from the first part of each approximate functional equation and the piece from the second part of each approximate functional equation. In the other two possible pieces the terms involving the X-function cannot be non-oscillatory. (If the degrees of the two L-functions were different then these pieces could potentially contribute. Or, if we were concentrating on a different aspect, say the q- or level-aspect, them we would have more pieces contributing.)

Recall that $s = 1/2 + it$. Then, the first of our contributing pieces arises from

$$\int_0^T \sum_{h,k,m,n} \left(\frac{kn}{hm}\right)^{it} \frac{a_m b_n a'_h b'_k}{m^{1/2+\alpha} n^{1/2+\beta} h^{1/2+\gamma} k^{1/2+\delta}}\, dt \tag{1}$$

and the second from

$$\int_0^T \epsilon_1\epsilon_2 X_1(s+\alpha) X_2(1-s+\beta) \sum_{h,k,m,n} \left(\frac{km}{hn}\right)^{it} \frac{\overline{a_m b_n} a'_h b'_k}{m^{1/2-\alpha} n^{1/2-\beta} h^{1/2+\gamma} k^{1/2+\delta}}\, dt \tag{2}$$

The non-oscillating terms in the first piece are those with $hm = kn$. In the second piece we have

$$X_1(s+\alpha)X_2(1-s+\beta) = Q_1^{-2\alpha} Q_2^{-2\beta} (q_2/q_1)^{it} \left(\frac{t}{2\pi}\right)^{-d(\alpha+\beta)} \left(1 + O\left(\frac{1}{t}\right)\right)$$

so that the non-oscillatory terms are those for which $q_2 km = q_1 hn$.

For simplicity, we now assume that $Q_1 = Q_2$, ie. that both L-functions have the same level. This assumption does not change our analysis much; if the L-functions had different levels we would have some relatively innocuous arithmetic factors depending on q_1 and q_2 entering the picture.

So, now we consider the first expression, but with summing over all h, m, k, n subject to $hm = kn$ and likewise the second expression but with $km = hn$. All the variables in both sums extend to infinity ; we assume in each case separately that the variables $\alpha, \beta, \gamma, \delta$ are such that these are convergent sums. (Note: the requirements on the domains of these variables are different in the two cases and might not have any overlap.)

Using the Euler product, we find that

$$\sum_{hm=kn} \frac{a_m b_n a'_h b'_k}{m^{1/2+\alpha} n^{1/2+\beta} h^{1/2+\gamma} k^{1/2+\delta}} = \prod_p \sum_{h+m=k+n} \frac{a_{p^m} b_{p^n} a'_{p^h} b'_{p^k}}{p^{m(1/2+\alpha)+n(1/2+\beta)+h(1/2+\gamma)+k(1/2+\delta)}}$$

$$= \prod_p \left(1 + \frac{a_p b_p}{p^{1+\alpha+\beta}} + \frac{a_p b'_p}{p^{1+\alpha+\delta}} + \frac{a'_p b_p}{p^{1+\gamma+\beta}} + \frac{a'_p b'_p}{p^{1+\gamma+\delta}} + \dots\right) \tag{3}$$

Now, $a'_p = -a_p$ and $b'_p = -b_p$. Therefore the above is

$$= \frac{L(1+\alpha+\beta)L(1+\gamma+\delta)}{L(1+\alpha+\delta)L(1+\beta+\gamma)} A(\alpha,\beta;\gamma,\delta)$$

where A is an Euler product which is absolutely convergent as long as $\Re\alpha, \Re\beta, \Re\gamma, \Re\delta > -1/2$; and where

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n b_n}{n^s}; \tag{4}$$

this function should have meromorphic continuation to the whole complex plane with at most finitely many poles in $\Re s > 1/2$. Similarly, for the second term, we find that

$$\sum_{km=hn} \frac{\overline{a_m b_n} a'_h b'_k}{m^{1/2-\alpha} n^{1/2-\beta} h^{1/2+\gamma} k^{1/2+\delta}} = \prod_p \left(1 + \frac{\overline{a_p b_p}}{p^{1-\alpha-\beta}} + \frac{\overline{a_p} a'_p}{p^{1-\alpha+\gamma}} + \frac{a'_p b'_p}{p^{1+\gamma+\delta}} + \frac{\overline{b_p} b'_p}{p^{1-\beta+\delta}} + \dots\right)$$

Again, using that $a'_p = -a_p$ and $b'_p = -b_p$ and

$$Z_1(s) = \sum_{m=1}^{\infty} \frac{|a_m|^2}{m^s} \qquad Z_2(s) = \sum_{n=1}^{\infty} \frac{|b_n|^2}{n^s}, \tag{5}$$

we have that the above is

$$= \frac{\overline{L}(1-\alpha-\beta)L(1+\gamma+\delta)}{Z_1(1-\alpha+\gamma)Z_2(1-\beta+\delta)} B(\alpha,\beta;\gamma,\delta)$$

where B is an Euler product which is absolutely convergent for $|\alpha|, |\beta|, |\gamma|, |\delta| < 1/2$. In general Z_1 and Z_2 will have meromorphic continuations to all of $\mathbf{C}$ and will have simple poles at $s = 1$ provided that L_1 and L_2 are primitive (i.e. cannot be factored into two L-functions with similar properties); let R_1 and R_2 denote the residues at $s = 1$ of Z_1 and Z_2.

Conjecture 1. *For* $0 < |\alpha|, |\beta|, |\gamma|, |\delta| < 1/2$, *we have*

$$R(\alpha,\beta;\gamma,\delta) = \int_0^T \left(\frac{L(1+\alpha+\beta)L(1+\gamma+\delta)}{L(1+\alpha+\delta)L(1+\beta+\gamma)} A(\alpha,\beta;\gamma,\delta) + \epsilon_1\epsilon_2 \left(\frac{t}{2\pi}\right)^{-d(\alpha+\beta)} \frac{\overline{L}(1-\alpha-\beta)L(1+\gamma+\delta)}{Z_1(1-\alpha+\gamma)Z_2(1-\beta+\delta)} B(\alpha,\beta;\gamma,\delta) \right) dt + O(T^{1/2+\epsilon})$$

In order to find our formula for the pair correlation, we need to differentiate both sides of the above formula with respect to α and β and then set $\gamma = \alpha$ and $\delta = \beta$. We note by equation (3) that

$$A(\alpha,\beta;\alpha,\beta) = 1.$$

Also,

$$\frac{d^2}{d\alpha d\beta}A(\alpha,\beta;\gamma,\delta)\Big|_{\substack{\gamma=\alpha\\ \delta=\beta}} = \sum_n \frac{\Lambda_{L_1}(n)\Lambda_{L_2}(n)}{n^{1+\alpha+\beta}}$$

where

$$\frac{L_i'}{L_i}(s) = -\sum_{n=1}^{\infty}\frac{\Lambda_{L_i}(n)}{n^s}.$$

Also, if $\gamma=\alpha$ and $\delta=\beta$, then

$$\sum_{km=hn}\frac{\overline{a_m b_n} a_h' b_k'}{m^{1/2-\alpha}n^{1/2-\beta}h^{1/2+\gamma}k^{1/2+\delta}} = \prod_p \sum_{k+m=h+n}\frac{\overline{a_{p^m} b_{p^n}} a_{p^h}' b_{p^k}'}{p^{h(1+\alpha+\beta)-m(\alpha+\beta)+n}}$$

is a function of $\alpha+\beta$ from which it follows easily that $B(\alpha,\beta;\alpha,\beta)$ is a function of $\alpha+\beta$. Thus, we are led to

Conjecture 2.

$$\int_0^T \frac{L_1'}{L_1}(s+\alpha)\frac{L_2'}{L_2}(1-s+\beta)\ dt = \int_0^T \Bigg(\left(\frac{L'}{L}\right)'(1+\alpha+\beta)$$
$$+A_2(\alpha+\beta) + \epsilon_1\epsilon_2\left(\frac{t}{2\pi}\right)^{-d(\alpha+\beta)}\frac{\overline{L}(1-\alpha-\beta)L(1+\alpha+\beta)}{R_1R_2}B_1(\alpha+\beta)\Bigg)\ dt$$
$$+O(T^{1/2+\epsilon})$$

where R_1 and R_2 are the residues of Z_1 and Z_2 at $s=1$ and where L, Z_1, and Z_2 are defined in equations (4) and (5); further,

$$A_2(\eta) = \sum_{n=1}^{\infty}\frac{\Lambda_{L_1}(n)\Lambda_{L_2}(n)}{n^{1+\eta}} - \left(\frac{L'}{L}\right)'(1+\eta),$$

and

$$B_1(\alpha+\beta) = B(\alpha,\beta;\alpha,\beta),$$

Now we want to evaluate the sum

$$S(f) = \sum_{\gamma_1,\gamma_2<T} f(\gamma_1-\gamma_2)$$

for a suitable test function f, where γ_i is the ordinate of a zero of $L_i(s)$. Actually, it is convenient for us to assume that f is holomorphic in a strip of fixed width, say 1, around the real axis and rapidly decaying as the absolute value of the real part of the variable gets large; suppose also that f is real and positive on the real axis , and even; this class of functions is suitably rich to determine the desired information about the pair-correlation function. We rewrite the sum in question in terms of contour integrals. Let $1/2<a<b<1$ and let $\mathcal{C}_1$ be the positively oriented rectangular contour with corners $a, a+iT, 1-a+iT, 1-a$ and let $\mathcal{C}_2$ be the rectangular contour with corners $b, b+iT, 1-b+iT, 1-b$. Then

$$S(f) = \frac{1}{(2\pi i)^2}\int_{\mathcal{C}_1}\int_{\mathcal{C}_2}\frac{L_1'}{L_1}(z)\frac{L_2'}{L_2}(w)f(-i(z-w))\ dw\ dz;$$

the point, of course, is that the poles inside the contours are simple poles with residue 1 at the zeros $z = 1/2 + i\gamma_1$ of L_1 and $w = 1/2 + i\gamma_2$ of L_2. The integrals along the horizontal contours are small and may be ignored. Thus, we consider 4 double integrals. We consider each of the 4 double integrals separately; call them $I_1, \dots, I_4$ where I_1 has vertical parts a and b; I_2 has vertical parts $1-a$ and $1-b$; I_3 has vertical parts a and $1-b$ and I_4 has vertical parts $1-a$ and b. It is easy to see that $I_1 = O(T^\epsilon)$ just by moving the contours to the right of 1 and integrating term-by-term. For I_2, we use the functional equations $\frac{L_i'}{L_i}(s) = \frac{X_i'}{X_i}(s) - \frac{L_i'}{L_i}(1-s)$ for $s = w$ and $s = z$ and find similarly that

$$I_2 = \frac{1}{(2\pi)^2} \int_0^T \int_0^T \frac{X_1'}{X_1}(1/2+iu)\frac{X_2'}{X_2}(1/2+iv) f(u-v)\, du\, dv + O(T^\epsilon).$$

Using the fact that

$$\frac{X_i'}{X_i}(1/2+it) = d \log\frac{|t|}{2\pi}\left(1 + O\left(\frac{1}{|t|}\right)\right)$$

and that f is even, we see, after the substitution $u = v + \eta$, that

$$\begin{aligned} I_2 &= \frac{2d}{(2\pi)^2} \int_0^T \int_0^u \log\frac{u}{2\pi} \log\frac{v}{2\pi} f(u-v)\, du\, dv + O(T^\epsilon) \\ &= \frac{2d}{(2\pi)^2} \int_0^T f(\eta) \int_0^{T-\eta} \log\frac{v}{2\pi} \log\frac{v+\eta}{2\pi}\, dv\, d\eta + O(T^\epsilon). \end{aligned}$$

Let us assume that f satisfies

$$f(x) \ll \frac{1}{1+x^2} \tag{6}$$

for real x. Letting $v \to vT$ in the inner integral above, we have

$$I_2 = \frac{2d}{(2\pi)^2} T \int_0^T f(\eta) \int_0^{1-\frac{\eta}{T}} \log\frac{vT}{2\pi} \log\frac{vT+\eta}{2\pi}\, dv\, d\eta + O(T^\epsilon).$$

We may extend the upper limit of the inner integral to $v = 1$, introducing an error term of size $\ll \int \eta f(\eta) \log^2 T \ll \log^3 T$. We can also replace $\log(vT + \eta)$ by $\log vT$ with the same error term. Thus,

$$\begin{aligned} I_2 &= \frac{2d}{(2\pi)^2} T \int_0^T f(\eta) \int_0^1 \log^2\frac{vT}{2\pi}\, dv\, d\eta + O(T^\epsilon) \\ &= \frac{2d}{(2\pi)^2} \int_0^T f(\eta) \int_0^T \log^2\frac{v}{2\pi}\, dv\, d\eta + O(T^\epsilon) \\ &= \frac{d}{(2\pi)^2} \int_{-T}^T f(\eta) \int_0^T \log^2\frac{v}{2\pi}\, dv\, d\eta + O(T^\epsilon). \end{aligned}$$

Next we consider I_3. Letting $z = w + \eta$, it is

$$\begin{aligned} I_3 &= \frac{-1}{(2\pi i)^2} \int_a^{a+iT} \int_{1-b}^{1-b+iT} \frac{L_2'}{L_2}(w)\frac{L_1'}{L_1}(z) f(-i(z-w))\, dz\, dw \\ &= \frac{-1}{(2\pi i)^2} \int_{1-a-b-iT}^{1-a-b+iT} f(-i\eta) \int_{T_1}^{T_2} \frac{L_2'}{L_2}(a+it)\frac{L_1'}{L_1}(a+it+\eta)\, dt\, d\eta \end{aligned}$$

where $T_1 = \max\{0, -\Im\eta\}$ and $T_2 = \min\{T - \Im\eta, T\}$. We use the functional equation

$$\frac{L_1'}{L_1}(a+\eta+it) = \frac{X_1'}{X_1}(a+\eta+it) - \frac{L_1'}{L_1}(1-a-\eta-it).$$

The term with the X_1'/X_1 is small as is seen by moving the contour to the right. Thus, we see that

$$\begin{aligned} I_3 &= \frac{1}{(2\pi i)^2}\int_{1-a-b-iT}^{1-a-b+iT} f(-i\eta)\int_{T_1}^{T_2}\frac{L_2'}{L_2}(a+it)\frac{L_1'}{L_1}(1-a-it-\eta)\ dt\ d\eta + O(T^\epsilon) \\ &= \frac{1}{(2\pi i)^2}\int_{1-a-b-iT}^{1-a-b+iT} f(-i\eta)\int_{T_1}^{T_2}\frac{L_2'}{L_2}(s+(a-1/2))\frac{L_1'}{L_1}(1-s+(1/2-a-\eta))\ dt\ d\eta \\ &\quad + O(T^\epsilon) \end{aligned}$$

where $s = 1/2 + it$. By Conjecture 2, we have

$$\begin{aligned} I_3 &= \frac{1}{(2\pi i)^2}\int_{1-a-b-iT}^{1-a-b+iT} f(-i\eta)\int_{T_1}^{T_2}\left(\frac{L'}{L}\right)'(1-\eta) \\ &\quad + A_2(-\eta) + \epsilon_1\epsilon_2\left(\frac{t}{2\pi}\right)^{d\eta}\frac{\overline{L}(1+\eta)L(1-\eta)}{R_1R_2}B_1(-\eta)\Bigg)\ dt\ d\eta + O(T^{1/2+\epsilon}) \end{aligned}$$

Let $\delta = a+b-1$ and let $g(-\eta, t)$ be the integrand in the second integral above. We can extend the range of the inner integration, much as we did for the I_2 integral to the interval $[0, T]$ with an error term of size $\ll T^\epsilon \int_\eta |\eta||f(\eta)| \ll T^\epsilon$. Thus, we obtain

$$I_3 = \frac{1}{(2\pi)^2}\int_0^T\int_{-\delta-iT}^{-\delta+iT} f(-i\eta)g(-\eta,t)\ d\eta\ dt + O(T^{1/2+\epsilon}).$$

Now we consider I_4. Again letting $z = w + \eta$, we have

$$\begin{aligned} I_4 &= \frac{1}{(2\pi i)^2}\int_{1-a}^{1-a+iT}\int_b^{b+iT}\frac{L_2'}{L_2}(w)\frac{L_1'}{L_1}(z)f(-i(z-w))\ dz\ dw \\ &= \frac{1}{(2\pi i)^2}\int_{a+b-1-iT}^{a+b-1+iT} f(-i\eta)\int_{T_1}^{T_2}\frac{L_2'}{L_2}(1-a+it)\frac{L_1'}{L_1}(1-a+it+\eta)\ dt\ d\eta \end{aligned}$$

We use the functional equation

$$\frac{L_2'}{L_2}(1-a+it) = \frac{X_2'}{X_2}(1-a+it) - \frac{L_2'}{L_2}(a-it).$$

Again, the contribution of the X_2'/X_2 term is negligible. Thus,

$$\begin{aligned} I_4 &= \frac{1}{(2\pi i)^2}\int_{a+b-1-iT}^{a+b-1+iT} f(-i\eta)\int_{T_1}^{T_2}\frac{L_2'}{L_2}(a-it)\frac{L_1'}{L_1}(1-a+it+\eta)\ dt\ d\eta + O(T^\epsilon) \\ &= \frac{1}{(2\pi i)^2}\int_{a+b-1-iT}^{a+b-1+iT} f(-i\eta)\int_{T_1}^{T_2}\frac{L_2'}{L_2}(1-s+(a-1/2))\frac{L_1'}{L_1}(s+(1/2-a+\eta))\ dt\ d\eta \\ &\quad + O(T^\epsilon) \end{aligned}$$

Now, by Conjecture 2,

$$I_4 = \frac{1}{(2\pi i)^2}\int_{a+b-1-iT}^{a+b-1+iT} f(-i\eta)\int_{T_1}^{T_2}\left(\frac{L'}{L}\right)'(1+\eta) \\ +A_2(\eta)+\epsilon_1\epsilon_2\left(\frac{t}{2\pi}\right)^{-d\eta}\frac{\overline{L}(1-\eta)L(1+\eta)}{R_1R_2}B_1(\eta)\Big)\ dt\ d\eta + O(T^{1/2+\epsilon})$$

Using the notation introduced after the calculation of I_3, and again extending the range of the integration in the inner integral, we can write the expression for I_4 as

$$I_4 = \frac{1}{(2\pi)^2}\int_0^T\int_{\delta-iT}^{\delta+iT} f(-i\eta)g(\eta,t)\ d\eta\ dt + O(T^\epsilon).$$

Combining this with what we found for I_3 we have, after a change of variables,

$$I_3+I_4 = \frac{2}{(2\pi)^2}\int_0^T\int_{\delta-iT}^{\delta+iT} f(i\eta)g(\eta,t)\ d\eta\ dt + O(T^\epsilon)$$

Now

$$g(\eta) = \left(\frac{L'}{L}\right)'(1+\eta)+A_2(\eta)+\epsilon_1\epsilon_2\left(\frac{t}{2\pi}\right)^{-d\eta}\frac{\overline{L}(1-\eta)L(1+\eta)}{R_1R_2}B_1(\eta)\Big)$$

We move the path of integration in η to the imaginary axis from $-T$ to T. Then, combining our expressions for $I_1,\ldots,I_4$, and changing η into ir we have

Theorem 1. *Assuming Conjecture 2,*

$$\sum_{\gamma_1,\gamma_2\le T} f(\gamma_1-\gamma_2) = \frac{1}{(2\pi)^2}\int_0^T\int_{-T}^{T} f(r)\Big(d\log^2\frac{t}{2\pi}+2\Big(\Big(\frac{L'}{L}\Big)'(1+ir) \\ +\epsilon_1\epsilon_2\left(\frac{t}{2\pi}\right)^{-idr}(R_1R_2)^{-1}L(1-ir)\overline{L}(1+ir)B_1(ir)+A_2(ir)\Big)\Big)\ dr\Big)\ dt + O(T^{1/2+\epsilon}).$$

Thus, the pair correlation kernel is

$$K(r):=\frac{1}{2\pi^2}\int_0^T\Big(\frac{d}{2}\log^2\frac{t}{2\pi}+\left(\frac{L'}{L}\right)'(1+ir) \\ +\epsilon_1\epsilon_2\left(\frac{t}{2\pi}\right)^{-idr}(R_1R_2)^{-1}L(1-ir)\overline{L}(1+ir)B_1(ir)+A_2(ir)\Big)\ dt.$$

2. Example: Dirichlet L-functions

Let us first consider the example with $d=1$. Then $L_1(s)=L(s,\chi_1)$ and $L_2(s)=L(s,\chi_2)$ where χ_1 and χ_2 are primitive Dirichlet characters with modulus q. We allow the possibility that $\chi_1=\chi_2$ or $\chi_1=\overline{\chi_2}$ and also the possibility that $q=1$ in which case both $L_i(s)=\zeta(s)$. We have

$$L(s)=\sum_{m=1}^{\infty}\frac{\chi_1\chi_2(m)}{m^s}=L(s,\chi)$$

where $\chi = \chi_1\chi_2$ is a character (not necessarily primitive) modulo q. Also,

$$Z_1(s) = Z_2(s) = \sum_{m=1}^{\infty} \frac{|\chi_i(m)|}{m^s} = \prod_{p\nmid q}\left(1-\frac{1}{p^s}\right)^{-1} = \prod_{p|q}\left(1-\frac{1}{p^s}\right)\zeta(s)$$

so that

$$R_1 = R_2 = \prod_{p|q}\left(1-\frac{1}{p}\right) = \frac{\phi(q)}{q}$$

where ϕ is Euler's phi-function. In this example, the sum (3) may be calculated easily since $\mu(p^h) = 0$ for $h > 1$; so by splitting into four cases we find that

$$\begin{aligned}\sum_{hm=kn} \frac{\chi_1(mh)\mu(h)\chi_2(nk)\mu(k)}{m^{1/2+\alpha}n^{1/2+\beta}h^{1/2+\gamma}k^{1/2+\delta}} &= \prod_p \sum_{m=0}^{\infty} \frac{\chi(p^m)}{p^{m(1+\alpha+\beta)}}\left(1-\frac{\chi(p)}{p^{1+\beta+\gamma}}-\frac{\chi(p)}{p^{1+\alpha+\delta}}+\frac{\chi(p)}{p^{1+\gamma+\delta}}\right)\\ &= \frac{L(1+\alpha+\beta)L(1+\gamma+\delta)}{L(1+\alpha+\delta)L(1+\beta+\gamma)}A(\alpha,\beta;\gamma,\delta)\end{aligned}$$

where

$$A(\alpha,\beta;\gamma,\delta) = \prod_p \frac{\left(1-\frac{\chi(p)}{p^{1+\beta+\gamma}}-\frac{\chi(p)}{p^{1+\alpha+\delta}}+\frac{\chi(p)}{p^{1+\gamma+\delta}}\right)\left(1-\frac{\chi(p)}{p^{1+\gamma+\delta}}\right)}{\left(1-\frac{\chi(p)}{p^{1+\beta+\gamma}}\right)\left(1-\frac{\chi(p)}{p^{1+\alpha+\delta}}\right)}$$

Then

$$A_2(\alpha+\beta) = \frac{d}{d\alpha}\frac{d}{d\beta}A(\alpha,\beta;\gamma,\delta)\Big|_{\substack{\gamma=\alpha\\ \delta=\beta}} = -\sum_p\left(\frac{\chi(p)\log p}{p^{1+\alpha+\beta}-\chi(p)}\right)^2.$$

Similarly, splitting into four cases according to the values of p^h and p^k, we calculate that

$$\begin{aligned}\sum_{km=hn} \frac{\overline{\chi_1(m)\chi_2(n)}\mu(h)\chi_1(h)\mu(k)\chi_2(k)}{m^{1/2-\alpha}n^{1/2-\beta}h^{1/2+\gamma}k^{1/2+\delta}} &= \prod_p \sum_{k+m=h+n} \frac{\overline{\chi_1(p^m)\chi_2(p^n)}\mu(p^h)\mu(p^k)\chi_1(p^h)\chi_2(p^k)}{p^{m(1/2-\alpha)+n(1/2-\beta)+h(1/2+\gamma)+k(1/2+\delta)}}\\ &= \prod_p \sum_{m=0}^{\infty} \frac{\overline{\chi}(p^m)}{p^{m(1-\alpha-\beta)}}\left(1-\frac{|\chi_1(p)|}{p^{1-\alpha+\gamma}}-\frac{|\chi_2(p)|}{p^{1-\beta+\delta}}+\frac{\chi(p)}{p^{1+\gamma+\delta}}\right)\\ &= \frac{L(1-\alpha-\beta,\overline{\chi})L(1+\gamma+\delta,\chi)}{L(1-\alpha+\gamma,\chi_0)L(1-\beta+\delta,\chi_0)}B(\alpha,\beta;\gamma,\delta)\end{aligned}$$

where χ_0 is the principal character modulo q and

$$B(\alpha,\beta;\gamma,\delta) = \prod_p \frac{\left(1-\frac{\chi_0(p)}{p^{1-\alpha+\gamma}}-\frac{\chi_0(p)}{p^{1-\beta+\delta}}+\frac{\chi(p)}{p^{1+\gamma+\delta}}\right)\left(1-\frac{\chi(p)}{p^{1+\gamma+\delta}}\right)}{\left(1-\frac{\chi_0(p)}{p^{1-\alpha+\gamma}}\right)\left(1-\frac{\chi_0(p)}{p^{1-\beta+\delta}}\right)}$$

Thus,

$$B_1(\alpha+\beta) = B(\alpha,\beta;\alpha,\beta) = \prod_p \frac{(1-\frac{\chi(p)}{p^{1+\alpha+\beta}})(1-\frac{2\chi_0(p)}{p}+\frac{\chi(p)}{p^{1+\alpha+\beta}})}{(1-\frac{\chi_0(p)}{p})^2}$$

Theorem 2. *Assuming conjecture 2 for Dirichlet L-functions, if χ_1 and χ_2 are primitive characters modulo q and $\chi = \chi_1\chi_2$, then*

$$\sum_{\substack{\gamma_1,\gamma_2\le T\\ L(1/2+i\gamma_j,\chi_j)=0}} f(\gamma_1-\gamma_2) = \frac{1}{(2\pi)^2}\int_0^T \Bigg(2\pi f(0)\delta_{\chi_1,\chi_2}\log\frac{t}{2\pi} + \int_{-T}^{T} f(r)\Bigg(\log^2\frac{t}{2\pi} + 2\Bigg(\left(\frac{L'}{L}\right)'(1+ir,\chi)$$

$$+\epsilon_1\epsilon_2\frac{q^2}{\phi(q)^2}\left(\frac{t}{2\pi}\right)^{-ir} L(1-ir,\overline{\chi})L(1+ir,\chi)B_1(ir)+A_2(ir)\Bigg)\Bigg)\,dr\Bigg)\,dt + O(T^{1/2+\epsilon});$$

where

$$B_1(\eta)=\prod_p \frac{(1-\frac{\chi(p)}{p^{1+\eta}})(1-\frac{2\chi_0(p)}{p}+\frac{\chi(p)}{p^{1+\eta}})}{(1-\frac{\chi_0(p)}{p})^2},$$

and

$$A_2(\eta) = -\sum_p\left(\frac{\chi(p)\log p}{p^{1+\eta}-\chi(p)}\right)^2$$

3. Example: Modular L-functions

Now we assume that both of our L-functions are of degree 2. In this case, we have

$$L_1(s)=\prod_p\left(1-\frac{\alpha_p}{p^s}\right)^{-1}\left(1-\frac{\alpha_p^*}{p^s}\right)^{-1}$$

and

$$L_2(s)=\prod_p\left(1-\frac{\beta_p}{p^s}\right)^{-1}\left(1-\frac{\beta_p^*}{p^s}\right)^{-1}.$$

Then

$$\frac{L_1'}{L_1}(s) = -\sum_{n=1}^{\infty}\frac{\Lambda_{L_1}(n)}{n^s} = -\sum_p \log p\sum_{k=1}^{\infty}\frac{\alpha_p^k+(\alpha_p^*)^k}{p^{ks}}$$

and similarly for L_2. It is well known that (the Rankin-Selberg convolution is)

$$L(s)=\sum_{n=1}^{\infty}\frac{a_nb_n}{n^s}=\prod_p\left(1-\frac{\alpha_p\alpha_p^*\beta_p\beta_p^*}{p^{2s}}\right)\left(1-\frac{\alpha_p\beta_p}{p^s}\right)^{-1}\left(1-\frac{\alpha_p\beta_p^*}{p^s}\right)^{-1}\left(1-\frac{\alpha_p^*\beta_p}{p^s}\right)^{-1}\left(1-\frac{\alpha_p^*\beta_p^*}{p^s}\right)^{-1}.$$

From these expressions, we see that

$$\left(\frac{L'}{L}\right)'(s) = \sum_p(\log p)^2\sum_{k=1}^{\infty}k\left(\frac{(\alpha_p\beta_p)^k+(\alpha_p\beta_p^*)^k+(\alpha_p^*\beta_p)^k+(\alpha_p^*\beta_p^*)^k}{p^{ks}} - 4\frac{(\alpha_p\alpha_p^*\beta_p\beta_p^*)^k}{p^{2ks}}\right)$$

and

$$\sum_{n=1}^{\infty}\frac{\Lambda_{L_1}(n)\Lambda_{L_2}(n)}{n^s} = \sum_p(\log p)^2\sum_{k=1}^{\infty}\left(\frac{(\alpha_p\beta_p)^k+(\alpha_p\beta_p^*)^k+(\alpha_p^*\beta_p)^k+(\alpha_p^*\beta_p^*)^k}{p^{ks}}\right).$$

Thus,

$$A_2(\eta) = \sum_p (\log p)^2 \left(\sum_{k=2}^\infty (1-k) \frac{(\alpha_p\beta_p)^k + (\alpha_p\beta_p^*)^k + (\alpha_p^*\beta_p)^k + (\alpha_p^*\beta_p^*)^k}{p^{k(1+\eta)}} + \sum_{k=1}^\infty 4k \frac{(\alpha_p\alpha_p^*\beta_p\beta_p^*)^k}{p^{2k(1+\eta)}} \right)$$

Now if $b_n = \overline{a_n}$, then we may take $\beta_p = \overline{\alpha_p}$ and $\beta_p^* = \overline{\alpha_p^*}$ so that by the above formula (recall that $|\alpha_p| = 0$ or 1 so that $|\alpha_p|^2 = |\alpha_p|$) we have

$$Z_1(s) = \sum_{n=1}^\infty \frac{|a_n|^2}{n^s} = \prod_p \left(1 - \frac{|\alpha_p\alpha_p^*|}{p^{2s}}\right) \left(1 - \frac{|\alpha_p|}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p\overline{\alpha_p^*}}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p^*\overline{\alpha_p}}{p^s}\right)^{-1} \left(1 - \frac{|\alpha_p^*|}{p^s}\right)^{-1};$$

and $Z_2(s)$ is similar, but with βs.

To simplify things a little, let us now assume that our L-functions both arise from cusp forms in $S_k(\Gamma_0(q))$. In this case, the level is the same for both L-functions and the coefficients a_n and b_n are real. Then, for any prime $p \nmid q$ it is the case that $\alpha_p^* = 1/\alpha_p$ and similarly for β_p. Moreover, if $p \mid q$ then we may take $\alpha_p^* = 0$ and $\alpha_p = -1, 0,$ or 1.

The symmetric square L-functions are defined by

$$L_1(\mathrm{sym}^2, s) = \prod_p \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p\alpha_p^*}{p^s}\right)^{-1} \left(1 - \frac{(\alpha_p^*)^2}{p^s}\right)^{-1}$$

and similarly for L_2. These functions are entire. It is easy to check that

$$Z_1(s) = \frac{\zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right) L_1(\mathrm{sym}^2, s)}{\zeta(2s) \prod_{p|q} \left(1 - \frac{1}{p^{2s}}\right)}$$

so that

$$R_1 = \frac{\prod_{p|q} \left(1 - \frac{1}{p}\right) L_1(\mathrm{sym}^2, 1)}{\zeta(2) \prod_{p|q} \left(1 - \frac{1}{p^2}\right)};$$

similarly for R_2.