AUTOCORRELATION OF RATIOS OF CHARACTERISTIC POLYNOMIALS AND OF L-FUNCTIONS

by

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PART I
RANDOM MATRIX PREDICTIONS FOR RATIOS

1. Introduction

1.1. The statement. — Here for completeness we write down what is to be proved in these notes. Let $\Lambda_U(z)$ be the characteristic polynomial of a unitary matrix $U \in \mathrm{U}(N)$:

$$\Lambda_U(z) = \mathrm{Det}(\mathrm{Id} - zU^*) \ .$$

For $i = 1, \dots, n$ let α_i, β_i, γ_i, and δ_i be complex numbers in the range $\mathfrak{Re}\,\gamma_i < 0 < \mathfrak{Re}\,\delta_i$. Let the symmetric group S_{2n} act by permutations of the set

$$H := \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n\} \ ,$$

and take $\mathrm{S}_n \times \mathrm{S}_n$ to be the subgroup of S_{2n} which permutes the α_i's separately from the β_i's. Define $\mathbf{S}$ to be the operator that symmetrizes a $(\mathrm{S}_n \times \mathrm{S}_n)$-invariant function f by summing over cosets:

$$\mathbf{S} f(H) := \sum_{[w] \in \mathrm{S}_{2n}/(\mathrm{S}_n \times \mathrm{S}_n)} f(w^{-1} \cdot H) \ .$$

Theorem 1.1. — *If dU is the Haar measure of $\mathrm{U}(N)$ normalized by $\int_{\mathrm{U}(N)} dU = 1$, the following is true for all $N \in \mathbb{N}$:*

$$\int\limits_{\mathrm{U}(N)} \prod_{i=1}^{n} \frac{\Lambda_U(e^{\alpha_i})\Lambda_U(e^{\beta_i})}{\Lambda_U(e^{\gamma_i})\Lambda_U(e^{\delta_i})} dU = \mathbf{S}\left(e^{N\sum_{i=1}^{n}(\beta_i - \delta_i)} \prod_{j,k=1}^{n} \frac{(1-e^{\alpha_j-\delta_k})(1-e^{\gamma_j-\beta_k})}{(1-e^{\alpha_j-\beta_k})(1-e^{\gamma_j-\delta_k})} \right) \ .$$

Remark. — Notice that the integrand on the left-hand side can be rewritten as

$$\prod_{i=1}^{n} \frac{\Lambda_U(\mathrm{e}^{\alpha_i})\Lambda_U(\mathrm{e}^{\beta_i})}{\Lambda_U(\mathrm{e}^{\gamma_i})\Lambda_U(\mathrm{e}^{\delta_i})} = \mathrm{e}^{N\sum_{i=1}^{n}(\beta_i-\delta_i)} \prod_{i=1}^{n} \frac{\Lambda_U(\mathrm{e}^{\alpha_i})\Lambda_{U^*}(\mathrm{e}^{-\beta_i})}{\Lambda_U(\mathrm{e}^{\gamma_i})\Lambda_{U^*}(\mathrm{e}^{-\delta_i})} .$$

Remark. — The orthogonal polynomial method of Baik, Deift and Strahov [**BDS**] apparently yields an answer for ratios only in the restricted range $N \geq n$. However, we claim that the statement of Theorem 1.1 is true in the full range $N \geq 1$.

1.2. Some basic definitions of superanalysis. —

- Supermatrix:

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} .$$

  The matrix entries of M_{11} and M_{22} are commuting variables (complex numbers), the matrix entries of M_{12} and M_{21} are anticommuting variables (i.e. generators of an exterior algebra).
- Supertrace:

$$\mathrm{STr}\, M = -\mathrm{Tr}\, M_{11} + \mathrm{Tr}\, M_{22} .$$

- Superdeterminant:

$$\mathrm{SDet}\, M = \frac{\mathrm{Det}\, M_{22}}{\mathrm{Det}(M_{11} - M_{12}M_{22}^{-1}M_{21})} .$$

  SDet and STr are related by

$$\mathrm{SDet} = \exp \circ \mathrm{STr} \circ \ln ,$$

  whenever SDet exists.
- The tensor product of a supermatrix M with an ordinary matrix U is again a supermatrix:

$$M \otimes U := \begin{pmatrix} M_{11} \otimes U & M_{12} \otimes U \\ M_{21} \otimes U & M_{22} \otimes U \end{pmatrix} .$$

Note that the reciprocal of a superdeterminant,

$$\mathrm{SDet}^{-1}(M) = \frac{\mathrm{Det}(M_{11} - M_{12}M_{22}^{-1}M_{21})}{\mathrm{Det}(M_{22})} ,$$

exists whenever the block M_{22} possesses an inverse.

1.3. The good holomorphic object to consider. — For $U \in \mathrm{U}(N)$ with Haar measure dU (normalized by $\int dU = 1$), consider the function χ defined as the integral

$$\chi(M) = \int_{\mathrm{U}(N)} \mathrm{SDet}^{-1}(\mathrm{Id} - M \otimes U^*)\, dU \,,$$

which is guaranteed to make sense if $\mathrm{Id} - M_{22} \otimes U^*$ has an inverse for all $U \in \mathrm{U}(N)$, i.e., if the spectrum of M_{22} avoids the unit circle.

To see why $\chi(M)$ is a useful object, let M be a diagonal supermatrix:

$$\begin{aligned} &\chi\Big(\mathrm{diag}(\mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_n}, \mathrm{e}^{\beta_1}, \ldots, \mathrm{e}^{\beta_n}, \mathrm{e}^{\gamma_1}, \ldots, \mathrm{e}^{\gamma_n}, \mathrm{e}^{\delta_1}, \ldots, \mathrm{e}^{\delta_n})\Big) \\ = \;& \int\limits_{\mathrm{U}(N)} \prod_{j=1}^{n} \frac{\mathrm{Det}(\mathrm{Id} - \mathrm{e}^{\alpha_j} U^*)\, \mathrm{Det}(\mathrm{Id} - \mathrm{e}^{\beta_j} U^*)}{\mathrm{Det}(\mathrm{Id} - \mathrm{e}^{\gamma_j} U^*)\, \mathrm{Det}(\mathrm{Id} - \mathrm{e}^{\delta_j} U^*)}\, dU \\ = \;& \mathrm{e}^{N \sum_j (\beta_j - \delta_j)} \int\limits_{\mathrm{U}(N)} \prod_{j=1}^{n} \frac{\Lambda_U(\mathrm{e}^{\alpha_j})\, \Lambda_{U^*}(\mathrm{e}^{-\beta_j})}{\Lambda_U(\mathrm{e}^{\gamma_j})\, \Lambda_{U^*}(\mathrm{e}^{-\delta_j})}\, dU \,. \end{aligned}$$

This is the function we are interested in.

The (super)function $M \mapsto \chi(M)$ is holomorphic, but only piecewise so, because of the singularities that occur when one or several of the eigenvalues of M_{22} hit the unit circle. All discussion will be carried out in one of the domains of holomorphicity, which is specified as follows.

Let $\mathrm{U}(n,n)$ be the noncompact pseudo-unitary group of $2n \times 2n$ matrices T with the property $T = s T^{-1^*} s$, where $s = \mathrm{diag}(1_n, -1_n)$. [Why $\mathrm{U}(n,n)$? The answer is that χ will be seen to be the character of a representation which is unitary for $\mathrm{U}(n,n)$.] Then let $\mathcal{U}^{(1)} := \mathrm{U}(2n)$, and define $\mathcal{U}^{(2)}$ to be the set of $2n \times 2n$ matrices of the form

$$\mathcal{U}^{(2)} := \{T\, \mathrm{diag}(A, D)\, T^{-1} \mid A, D \in \mathrm{Herm}(\mathbb{C}^n);\, 0 < A < 1 < D;\, T \in \mathrm{U}(n,n)\} \,.$$

Thus $\mathcal{U}^{(2)}$ is the orbit of $\mathrm{U}(n,n)$ (acting by conjugation) of the block-diagonal positive Hermitian matrices $\mathrm{diag}(A, D)$ where the eigenvalues of A are less than unity, and those of D are greater than unity. $\mathcal{U}^{(2)}$ is a $(2n)^2$-dimensional real-analytic submanifold of $\mathrm{GL}_{2n}(\mathbb{C})$. We shall study χ as a (super)function on the product manifold $\mathcal{U} := \mathcal{U}^{(1)} \times \mathcal{U}^{(2)}$, i.e., for $M_{11} \in \mathrm{U}(2n)$ and $M_{22} \in \mathcal{U}^{(2)}$. Of course, χ extends to a holomorphic function on a tubular complex neighborhood of $\mathcal{U}$.

As a function on that domain, χ has a number of strong properties which determine it completely and will be used to prove Theorem 1.1.

1.4. Outline of strategy. —

- Show that $\mathrm{SDet}^{-1}(\mathrm{Id} - M \otimes U)$ is a character in the tensor product of the oscillator (or Shale-Weil) representation of the metaplectic group with the spinor representation of the spin group.

- Using the fact that Howe pairs act without multiplicity in the oscillator-spinor representation, show that $\chi(M) = \int_{\mathrm{U}(N)} \mathrm{SDet}^{-1}(\mathrm{Id} - M \otimes U^*)\, dU$ is the character associated with an irreducible representation of the Lie superalgebra $\mathfrak{gl}_{2n|2n}$.

These statements will first be established for the simple case where M is a diagonal matrix. Afterwards we introduce the notion of Lie supergroup and explain how to make sense of the character $\chi(M)$ as a superfunction on the Lie supergroup $\mathrm{GL}(2n|2n)$. Then we use some basic results of supermanifold theory to compute $\chi(M)$. Outline:

- As an irreducible character, $M \mapsto \chi(M)$ is a joint eigenfunction of the ring of $\mathfrak{gl}_{2n|2n}$-invariant differential operators. Show that the eigenvalue is zero for the full ring.
- Using the explicit form of the (radial parts of the) invariant differential operators given by Berezin [**B**], verify that the claimed expression (Theorem 1.1) for χ is indeed a solution to the problem of finding a zero eigenfunction of all these operators.
- Prove uniqueness of the solution.

2. $\langle$Ratio$\rangle$ is a character

2.1. $\mathfrak{gl}_{n|n} \times \mathrm{U}(N)$ acting in the oscillator-spinor representation. — Given a complex vector space $V = \mathbb{C}^d$ with unitary structure $\langle \cdot, \cdot \rangle$, consider the exterior algebra $\wedge(V)$ and symmetric algebra $S(V)$. On the former we have the operations of exterior multiplication ε and inner multiplication ι:

$$\begin{aligned} \varepsilon(v): &\quad \wedge^k(V) \to \wedge^{k+1}(V) \quad (v \in V)\,, \\ \iota(f): &\quad \wedge^k(V) \to \wedge^{k-1}(V) \quad (f \in V^*)\,, \end{aligned}$$

on the latter the operations of (symmetric) multiplication μ and differentiation δ:

$$\begin{aligned} \mu(v): &\quad S^k(V) \to S^{k+1}(V) \quad (v \in V)\,, \\ \delta(f): &\quad S^k(V) \to S^{k-1}(V) \quad (f \in V^*)\,. \end{aligned}$$

The algebraic relations obeyed by these operators are the canonical anticommutation relations (CAR) on the exterior algebra (or "fermionic") side:

$$\begin{aligned} \varepsilon(v)\,\varepsilon(v') + \varepsilon(v')\,\varepsilon(v) &= 0\,, \\ \iota(f)\,\iota(f') + \iota(f')\,\iota(f) &= 0\,, \\ \iota(f)\,\varepsilon(v) + \varepsilon(v)\,\iota(f) &= f(v)\,, \end{aligned} \tag{1}$$

and the canonical commutation relations (CCR) on the symmetric (or "bosonic") side:

$$\begin{aligned} \mu(v)\,\mu(v') - \mu(v')\,\mu(v) &= 0\,, \\ \delta(f)\,\delta(f') - \delta(f')\,\delta(f) &= 0\,, \\ \delta(f)\,\mu(v) - \mu(v)\,\delta(f) &= f(v)\,. \end{aligned} \tag{2}$$

Supersymmetry enters when we take the tensor product of $\wedge(V)$ with $S(V)$ (called the oscillator-spinor module). The above operators act on $\wedge(V)\otimes S(V)$ in the natural way: the fermionic operators $\varepsilon(v)$ and $\iota(f)$ act on the tensor factor $\wedge(V)$, the bosonic operators $\mu(v)$ and $\delta(f)$ act on the tensor factor $S(V)$, and the bosonic and fermionic actions commute.

The unitary structure $\langle\cdot,\cdot\rangle$ of V induces a canonical unitary structure on the oscillator-spinor module $\wedge(V)\otimes S(V)$ in which ε is adjoint to ι and μ is adjoint to δ:

$$\varepsilon(v)^* = \iota(Cv)\ , \quad \mu(v)^* = \delta(Cv)\ ,$$

where $C : V \to V^*$ is defined by $Cv = \langle v,\cdot\rangle$.

Let now $V = \mathbb{C}^n\otimes\mathbb{C}^N$ (later we will double $n\to 2n$) and write

$$\mathcal{V} := \wedge(\mathbb{C}^n\otimes\mathbb{C}^N)\otimes S(\mathbb{C}^n\otimes\mathbb{C}^N)\ .$$

If $\{e_a\}_{a=1,\ldots,N}$ and $\{e_j\}_{j=1,\ldots,n}$ are orthonormal bases of $\mathbb{C}^N$ and $\mathbb{C}^n$ respectively, let $\{f_a\}_{a=1,\ldots,N}$ and $\{f_j\}_{j=1,\ldots,n}$ be the corresponding dual bases, i.e., $f_a = \langle e_a,\cdot\rangle$ and $f_i = \langle e_i,\cdot\rangle$. The Lie algebra $\mathfrak{u}_N$ of the unitary group $\mathrm{U}(N)$ is represented on $\mathcal{V}$ by

$$A \mapsto r(A) = \sum_{a,b;j} f_a(Ae_b)\big(\varepsilon(e_j\otimes e_a)\,\iota(f_j\otimes f_b) + \mu(e_j\otimes e_a)\,\delta(f_j\otimes f_b)\big)\ .$$

Using the CAR and CCR relations it is easy to check that this is indeed a representation:

$$[r(A),r(B)] = r\,([A,B]) \quad (A,B\in\mathfrak{u}_N)\ .$$

r exponentiates to a unitary representation (which we still denote by r) of $\mathrm{U}(N)$.

Aside from the $\mathrm{U}(N)$-representation, the oscillator-spinor module $\mathcal{V}$ carries a canonical representation of the Lie superalgebra $\mathfrak{gl}_{n|n}$. The latter is abstractly defined as the algebra (over $\mathbb{C}$) generated by operators $E_{ij}^{\sigma\tau}$ with $i,j\in\{1,\ldots,n\}$ and $\sigma,\tau\in\{1,2\}$ satisfying the Lie (super)bracket relations

$$\begin{aligned}[][E_{ij}^{\sigma\tau},E_{i'j'}^{\sigma'\tau'}] &:= E_{ij}^{\sigma\tau}E_{i'j'}^{\sigma'\tau'} - (-1)^{(\sigma+\tau)(\sigma'+\tau')}E_{i'j'}^{\sigma'\tau'}E_{ij}^{\sigma\tau} \\ &= E_{ij'}^{\sigma\tau'}\delta_{ji'}\delta^{\tau\sigma'} - (-1)^{(\sigma+\tau)(\sigma'+\tau')}E_{i'j}^{\sigma'\tau}\delta_{j'i}\delta^{\tau'\sigma}\ . \end{aligned} \tag{3}$$

These generators are represented by operators on the oscillator-spinor module $\mathcal{V}$ as

$$\begin{aligned} \rho(E_{ij}^{11}) &= \textstyle\sum_a \varepsilon(e_i\otimes e_a)\,\iota(f_j\otimes f_a)\ , \quad \rho(E_{ij}^{12}) = \textstyle\sum_a \varepsilon(e_i\otimes e_a)\,\delta(f_j\otimes f_a)\ , \\ \rho(E_{ij}^{21}) &= \textstyle\sum_a \mu(e_i\otimes e_a)\,\iota(f_j\otimes f_a)\ , \quad \rho(E_{ij}^{22}) = \textstyle\sum_a \mu(e_i\otimes e_a)\,\delta(f_j\otimes f_a)\ . \end{aligned}$$

Again this is easily checked to be a representation:

$$\rho([X,Y]) = [\rho(X),\rho(Y)] \quad (X,Y\in\mathfrak{gl}_{n|n})\ .$$

This representation is unitary for the Lie algebra $\mathfrak{u}_n\oplus\mathfrak{u}_n\subset\mathfrak{gl}_{n|n}$, where each summand $\mathfrak{u}_n$ is a copy of the unitary Lie algebra associated with the vector space $\mathbb{C}^n$. For that subalgebra the representation exponentiates to a unitary representation of $\mathrm{U}(n)\times\mathrm{U}(n)$.

Since $\mathrm{U}(N)$ acts on one factor of a tensor product $\mathbb{C}^n \otimes \mathbb{C}^N$ and $\mathfrak{gl}_{n|n}$ acts on the other, it is clear that the two actions commute. In fact, $\mathrm{U}(N)$ and $\mathfrak{gl}_{n|n}$ form a *dual pair* in the sense of R. Howe. This will become important later on.

2.2. Oscillator-spinor character. — Consider a diagonal $\mathrm{U}(N)$-matrix U,

$$U = \mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\varphi_1}, \ldots, \mathrm{e}^{\mathrm{i}\varphi_N}\right) ,$$

and a diagonal "parameter" matrix M:

$$M = \mathrm{diag}\left(\mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_n}, \mathrm{e}^{\gamma_1}, \ldots, \mathrm{e}^{\gamma_n}\right) .$$

These are represented by operators $r(U)$ and $\rho(M)$ on the oscillator-spinor module $\mathcal{V}$. We will now evaluate the character of the oscillator-spinor representation on their product, i.e.,

$$\chi(M,U) := \mathrm{STr}_{\mathcal{V}}\, \rho(M)\, r(U) .$$

Since the matrices U and M, as well as their representations $r(U)$ and $\rho(M)$, are completely diagonal, this is easily done by the rules

$$\mu(e_i)\delta(f_i)\, e_i^p = z_i \frac{\partial}{\partial z_i} z_i^p = p z_i^p \qquad (p = 0, 1, 2, \ldots)$$

on the symmetric side where the basis vector e_i is viewed as a complex variable z_i, and

$$\varepsilon(e_i)\iota(f_i)\, e_i^{\wedge q} = \zeta_i \frac{\partial}{\partial \zeta_i} \zeta_i^q = q \zeta_i^q \qquad (q = 0, 1)$$

on the alternating side where each e_i is regarded as a generator ζ_i of $\wedge(\mathbb{C}^n)$.

Here then is how the character is computed:

$$\begin{aligned} \chi(M,U) &= \mathrm{STr}_{\mathcal{V}}\, \mathrm{e}^{\sum_{j,a}(\alpha_j + \mathrm{i}\varphi_a)\varepsilon(e_j\otimes e_a)\iota(f_j\otimes f_a) + \sum_{j,a}(\gamma_j+\mathrm{i}\varphi_a)\mu(e_j\otimes e_a)\delta(f_j\otimes f_a)} \\ &= \prod_{j=1}^{n}\prod_{a=1}^{N}\sum_{q=0}^{1}\sum_{p=0}^{\infty}(-1)^q \mathrm{e}^{q(\alpha_j+\mathrm{i}\varphi_a)+p(\gamma_j+\mathrm{i}\varphi_a)} = \prod_{j=1}^{n}\prod_{a=1}^{N}\frac{1-\mathrm{e}^{\alpha_j+\mathrm{i}\varphi_a}}{1-\mathrm{e}^{\gamma_j+\mathrm{i}\varphi_a}} . \end{aligned}$$

The geometric series $\sum_{p=0}^{\infty} \mathrm{e}^{p\gamma} = (1-\mathrm{e}^{\gamma})^{-1}$ converges for $\mathfrak{Re}\,\gamma < 0$. Therefore, the oscillator-spinor character $\chi(M,U)$ exists for $M = \mathrm{diag}(\mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\gamma_n})$ if $\mathfrak{Re}\,\gamma_j < 0$ for all $j = 1, \ldots, n$. Since the $\mathrm{U}(N)$-action on $\mathcal{V}$ is invariantly defined and any element $U \in \mathrm{U}(N)$ can be brought to diagonal form by a suitable choice of basis, we have the following statement.

Proposition 2.1*. — The oscillator-spinor character* $\mathrm{STr}_{\mathcal{V}}\rho(M)r(U)$ *for* $V = \mathbb{C}^n \otimes \mathbb{C}^N$, $\mathcal{V} = \wedge(V) \otimes S(V)$, $U \in \mathrm{U}(N)$, $M = \mathrm{diag}(\mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_n}, \mathrm{e}^{\gamma_1}, \ldots, \mathrm{e}^{\gamma_n})$, *and* $\mathfrak{Re}\,\gamma_j < 0$ *(*$j = 1, \ldots, n$*), is a ratio of determinants:*

$$\mathrm{STr}_{\mathcal{V}}\, \rho(M)\, r(U) = \prod_{j=1}^{n} \frac{\mathrm{Det}(\mathrm{Id} - \mathrm{e}^{\alpha_j} U)}{\mathrm{Det}(\mathrm{Id} - \mathrm{e}^{\gamma_j} U)} .$$

2.3. Adjusting the normal ordering. — The case dealt with by Prop. 2.1 is not the one we are interested in. Doubling the number of parameters (by replacing $n \to 2n$), we want the case $\mathfrak{Re}\,\gamma_j < 0$ for $j = 1, \ldots, n$ and $\mathfrak{Re}\,\gamma_j > 0$ for $j = n+1, \ldots, 2n$; or equivalently, in the notation of Sect. 1.3,

$$\mathfrak{Re}\,\gamma_j < 0 < \mathfrak{Re}\,\delta_j \quad (j = 1, \ldots, n) .$$

In that case the geometric sums $\sum_{p=0}^{\infty} \mathrm{e}^{p\delta_j}$ do not exist.

At the level of the right-hand side of the formula in Prop. 2.1 the problem is repaired by the manipulation

$$\frac{\mathrm{Det}(\mathrm{Id} - \mathrm{e}^{\beta_j} U)}{\mathrm{Det}(\mathrm{Id} - \mathrm{e}^{\delta_j} U)} = \mathrm{e}^{N(\beta_j - \delta_j)} \frac{\mathrm{Det}(\mathrm{Id} - \mathrm{e}^{-\beta_j} U^*)}{\mathrm{Det}(\mathrm{Id} - \mathrm{e}^{-\delta_j} U^*)} , \tag{4}$$

as a result of which the denominator $\mathrm{Det}^{-1}(\mathrm{Id} - \mathrm{e}^{-\delta_j} U^*)$ expands into a convergent power series with respect to $\mathrm{e}^{-\delta_j} U^*$.

What's the corresponding manipulation on the left-hand side? The answer is that in all of the expressions for the $\mathrm{U}(N)$- and $\mathfrak{gl}_{2n|2n}$-generators we make the replacements

$$\begin{aligned} \delta(f_j \otimes f_a) &\longrightarrow \mu(e_j \otimes e_a) , \quad \mu(e_j \otimes e_a) \longrightarrow -\delta(f_j \otimes f_a) , \\ \iota(f_j \otimes f_a) &\longrightarrow \varepsilon(e_j \otimes e_a) , \quad \varepsilon(e_j \otimes e_a) \longrightarrow +\iota(f_j \otimes f_a) , \end{aligned} \tag{5}$$

for $j = n+1, \ldots, 2n$. (Note the minus sign in the bosonic sector!) This means, for example, that the subset of generators E_{ij}^{22} of $\mathfrak{gl}_{2n|2n}$ are now represented by

$$\begin{aligned} E_{ii'}^{22} &\mapsto \textstyle\sum_a \mu(e_i \otimes e_a)\,\delta(f_{i'} \otimes f_a) , \quad E_{ji'}^{22} \mapsto -\textstyle\sum_a \delta(f_j \otimes f_a)\,\delta(f_{i'} \otimes f_a) , \\ E_{ij'}^{22} &\mapsto \textstyle\sum_a \mu(e_i \otimes e_a)\,\mu(e_{j'} \otimes e_a) , \quad E_{jj'}^{22} \mapsto -\textstyle\sum_a \delta(f_j \otimes f_a)\,\mu(e_{j'} \otimes e_a) , \end{aligned}$$

where we use the convention that $i, i' \in \{1, \ldots, n\}$ and $j, j' \in \{n+1, \ldots, 2n\}$. In physics one would say that the definition of the normal ordering is adapted to a new vacuum.

The transformation, say ψ, effecting the replacements (5) is an automorphism of the CAR relations (1) and CCR relations (2). Since these relations determine the Lie (super)bracket of the $\mathfrak{u}_N$- and $\mathfrak{gl}_{2n|2n}$-representations on $\mathcal{V}$, the mapping ψ is also an automorphism of the latter two. Thus we get new $\mathrm{U}(N)$- and $\mathfrak{gl}_{2n|2n}$-representations $\tilde{r} := \psi \circ r$ and $\tilde{\rho} := \psi \circ \rho$.

The modified representation $\tilde{r}$ is still unitary for $\mathrm{U}(N)$. However, because of the sign change in the bosonic sector, the $\mathfrak{gl}_{2n|2n}$-representation $\tilde{\rho}$ no longer exponentiates to a unitary representation $\tilde{\rho}$ of $\mathrm{U}(2n) \times \mathrm{U}(2n)$; rather $\tilde{\rho}$ is unitary for $\mathrm{U}(2n) \times \mathrm{U}(n,n)$.

Proposition 2.2. — *Take V to be the unitary vector space $V = \mathbb{C}^{2n} \otimes \mathbb{C}^N$, and let $\mathcal{V} = \wedge(V) \otimes S(V)$ be the associated oscillator-spinor module. Let M be a diagonal matrix:*

$$M = \mathrm{diag}\left(\mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_n}, \mathrm{e}^{\beta_1}, \ldots, \mathrm{e}^{\beta_n}, \mathrm{e}^{\gamma_1}, \ldots, \mathrm{e}^{\gamma_n}, \mathrm{e}^{\delta_1}, \ldots, \mathrm{e}^{\delta_n}\right) ,$$

with complex parameters in the range $\mathfrak{Re}\,\gamma_j < 0 < \mathfrak{Re}\,\delta_j$ ($j = 1,\ldots,n$). If $\tilde{r}$ and $\tilde{\rho}$ are the $\mathrm{U}(N)$- and $\mathfrak{gl}_{2n|2n}$-representations on $\mathcal{V}$ defined above, the oscillator-spinor character evaluated on a pair (M,U) with $U \in \mathrm{U}(N)$ is

$$\mathrm{STr}_{\mathcal{V}}\, \tilde{\rho}(M)\, \tilde{r}(U) = \prod_{j=1}^{n} \mathrm{e}^{N(\beta_j-\delta_j)} \frac{\mathrm{Det}(\mathrm{Id}-\mathrm{e}^{\alpha_j}U)\,\mathrm{Det}(\mathrm{Id}-\mathrm{e}^{-\beta_j}U^*)}{\mathrm{Det}(\mathrm{Id}-\mathrm{e}^{\gamma_j}U)\,\mathrm{Det}(\mathrm{Id}-\mathrm{e}^{-\delta_j}U^*)} \,.$$

Proof. — Since the $\mathrm{U}(N)$-representation $\tilde{r}$ is invariantly defined, it suffices to verify the statement for the special case of diagonal $U = \mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\varphi_1},\ldots,\mathrm{e}^{\mathrm{i}\varphi_N}\right)$, where one has

$$\begin{aligned}&\mathrm{STr}_{\mathcal{V}}\, \tilde{\rho}(M)\, \tilde{r}(U) = \\ &\mathrm{STr}_{\mathcal{V}} \Big(\mathrm{e}^{\sum_{j,a}(\alpha_j+\mathrm{i}\varphi_a)\,\varepsilon(e_j\otimes e_a)\iota(f_j\otimes f_a)+\sum_{j,a}(\gamma_j+\mathrm{i}\varphi_a)\,\mu(e_j\otimes e_a)\delta(f_j\otimes f_a)} \\ &\qquad \times \mathrm{e}^{\sum_{j,a}(\beta_j+\mathrm{i}\varphi_a)\,\iota(f_{j+n}\otimes f_a)\varepsilon(e_{j+n}\otimes e_a)-\sum_{j,a}(\delta_j+\mathrm{i}\varphi_a)\,\delta(f_{j+n}\otimes f_a)\mu(e_{j+n}\otimes e_a)} \Big) \,.\end{aligned}$$

Evaluating the supertrace in a basis of alternating and symmetric powers of the vectors $e_j \otimes e_a$, one gets multiple geometric sums as before. For example, the symmetric powers $(e_{j+n}\otimes e_a)^p$ are eigenvectors of the operator $-\delta(f_{j+n}\otimes f_a)\mu(e_{j+n}\otimes e_a)$ with eigenvalues $-(p+1)$, which gives

$$\sum_{p=0}^{\infty} \mathrm{e}^{-(p+1)(\delta_j+\mathrm{i}\varphi_a)} = \frac{\mathrm{e}^{-\delta_j-\mathrm{i}\varphi_a}}{1-\mathrm{e}^{-\delta_j-\mathrm{i}\varphi_a}} \,.$$

The geometric series converge in the stated range for γ_j and δ_j. On collecting all factors one obtains

$$\mathrm{STr}_{\mathcal{V}}\, \tilde{\rho}(M)\, \tilde{r}(U) = \prod_{j=1}^{n} \mathrm{e}^{N\sum_j(\beta_j-\delta_j)} \prod_{a=1}^{N} \frac{(1-\mathrm{e}^{\alpha_j+\mathrm{i}\varphi_a})(1-\mathrm{e}^{-\beta_j-\mathrm{i}\varphi_a})}{(1-\mathrm{e}^{\gamma_j+\mathrm{i}\varphi_a})(1-\mathrm{e}^{-\delta_j-\mathrm{i}\varphi_a})} \,,$$

as claimed. □

By undoing the manipulation (4) and recalling the definition of the superdeterminant SDet, the statement of Prop. 2.2 can be cast in the following succinct form.

Corollary 2.3. —

$$\mathrm{STr}_{\mathcal{V}}\, \tilde{\rho}(M)\tilde{r}(U) = \mathrm{SDet}_{\mathbb{C}^{2n|2n}\otimes\mathbb{C}^N}(\mathrm{Id}-M\otimes U)^{-1} \,.$$

2.4. Using Howe duality. — In an early and insightful article "Remarks on Classical Invariant Theory" [**H**], R. Howe introduced and studied the pair of representations

$$\tilde{r} : \mathrm{U}(N) \to \mathrm{U}(\mathcal{V}) \,, \quad \text{and} \quad \tilde{\rho} : \mathfrak{gl}_{2n|2n} \to \mathfrak{gl}(\mathcal{V})$$

(as one in a triplet of families of such pairs, which are nowadays called Howe pairs). Howe showed that the isotypic component of $\mathcal{V}$ associated with the trivial $\mathrm{U}(N)$-representation (i.e., the space of $\mathrm{U}(N)$-invariant vectors in $\mathcal{V}$) is an *irreducible* representation space for $\mathfrak{gl}_{2n|2n}$. We denote this space by $\mathcal{V}_0$.

Consider now the $\mathrm{U}(N)$-average of $\mathrm{STr}_{\mathcal{V}}\,\tilde\rho(M)\,\tilde r(U)$ for M as defined in the statement of Prop. 2.2. Since integration of $\tilde r(U)$ over $\mathrm{U}(N)$ with Haar measure projects on the $\mathrm{U}(N)$-invariants and thus on the linear subspace $\mathcal{V}_0 \subset \mathcal{V}$, Howe's result on dual pairs immediately yields:

Proposition 2.4. — *The function $M \mapsto \chi(M)$ defined by*

$$\chi(M) = \int_{\mathrm{U}(N)} \mathrm{STr}_{\mathcal{V}}\,\tilde\rho(M)\,\tilde r(U)\,dU\ ,$$

is an irreducible character of $\mathfrak{gl}_{2n|2n}$,

$$\chi(M) = \mathrm{STr}_{\mathcal{V}_0}\,\tilde\rho(M)\ ,$$

where $(\mathcal{V}_0,\tilde\rho)$ is the irreducible $\mathfrak{gl}_{2n|2n}$-representation associated with the trivial $\mathrm{U}(N)$-representation on the oscillator-spinor module $\mathcal{V} = \wedge(\mathbb{C}^{2n}\otimes\mathbb{C}^N)\otimes S(\mathbb{C}^{2n}\otimes\mathbb{C}^N)$ by the Howe dual-pair correspondence.

Remark. — By Prop. 2.2 the function χ is identical to the function on the left-hand side of the statement of Theorem 1.1. Thus we have established that left-hand side to be an irreducible $\mathfrak{gl}_{2n|2n}$-character. This completes the first step of the programme outlined in Sect. 1.

As a first preparation for the proof of Theorem 1.1, consider the limiting case where all parameters $\gamma_1,\ldots,\gamma_n$ and $\delta_1,\ldots,\delta_n$ (i.e., those residing in the denominator of the ratio of characteristic polynomials) are removed from the picture by sending in $\mathrm{e}^{N\sum_j\delta_j}\chi$ the former to $-\infty$ and the latter to $+\infty$:

$$\mathrm{e}^{N\sum_j\delta_j}\chi \longrightarrow \int_{\mathrm{U}(n)} \mathrm{Det}^n(-U^*)\prod_{j=1}^n \mathrm{Det}(\mathrm{Id}-\mathrm{e}^{\alpha_j}U)\,\mathrm{Det}(\mathrm{Id}-\mathrm{e}^{\beta_j}U)\,dU := \chi_s\ . \tag{6}$$

Another way to think about this limit is to say that $\mathfrak{gl}_{2n|2n}$ is being restricted to the Lie algebra $\mathfrak{gl}_{2n}$ (with Cartan subalgebra parameterized by $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots\beta_n$), and we now consider the dual pair $\mathfrak{gl}_{2n}\times\mathrm{U}(N)$ acting on the spinor module $\mathcal{V}_s = \wedge(\mathbb{C}^{2n}\otimes\mathbb{C}^N)$. Let $\mathcal{V}_{s,0}\subset\mathcal{V}_0$ be the subspace of $\mathrm{U}(N)$-invariants in $\mathcal{V}_s$. The $\mathfrak{gl}_{2n|2n}$-representation $(\mathcal{V}_0,\tilde\rho)$ then restricts to the $\mathfrak{gl}_{2n}$-representation $(\mathcal{V}_{s,0},\tilde\rho_s)$. From Howe [**H**] we know the latter representation to be still irreducible. Its character χ_s can be computed by standard techniques (see, e.g., [Refs]), and we here record the result for later use.

Proposition 2.5. — *The function χ_s defined in (6) is the character of the irreducible $\mathfrak{gl}_{2n}$-representation $(\mathcal{V}_{s,0},\tilde\rho_s)$. If $W := \mathrm{S}_{2n}$ is the symmetric group permuting the $2n$ variables $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n$, and $W_0 := \mathrm{S}_n\times\mathrm{S}_n\subset\mathrm{S}_{2n}$ is the subgroup which permutes the α's and β's separately, this character is expressed by the formula*

$$\chi_s = \sum_{[w]\in W/W_0}\frac{\mathrm{e}^{N\sum_j w(\beta_j)}}{\prod_{k,l}(1-\mathrm{e}^{w(\alpha_k)-w(\beta_l)})}\ .$$

3. Background material from supermanifold theory

It remains to compute the character $M \mapsto \chi(M)$ and prove that it is given by the right-hand side of Theorem 1.1. For that purpose it is not prudent to adhere to the limited view of χ as a function of the diagonal matrices M. Rather, to get enough analytical control we are going to exploit the fact that χ extends to a function on the Lie supergroup $\mathrm{GL}(2n|2n)$. The desired statement then follows rather easily from some basic results about the analysis on supermanifolds.

Since supermanifold theory is not a widely known subject, we inject the present section to collect some of the necessary material for the convenience of the reader. The computation of χ is then carried out in Sect. 4.

3.1. The vector bundle underlying a Lie supergroup. — Lie supergroups have been defined and studied by Berezin [**B**]. Here we will review a simplified definition, which uses nothing but standard constructions in geometry.

We start from a *complex Lie superalgebra*, which in brief is a $\mathbb{Z}_2$-graded complex vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ equipped with a parity function $|\cdot|$ (i.e., $|X| = 0$ for $X \in \mathfrak{g}_0$ and $|X| = 1$ for $X \in \mathfrak{g}_1$) and with a bracket operation $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. In the case of a matrix Lie superalgebra $\mathfrak{g}$ (such as $\mathfrak{g} = \mathfrak{gl}$ or $\mathfrak{g} = \mathfrak{osp}$) where products XY make sense as matrix products, the bracket is defined by

$$[X,Y] := XY - (-1)^{|X||Y|} YX$$

for elements X, Y of definite parity, and is then extended to all of $\mathfrak{g}$ by linearity.

In the sequel we will be concerned only with the two classical Lie superalgebras, namely $\mathfrak{gl}$ and $\mathfrak{osp}$. To define $\mathfrak{g} = \mathfrak{osp}_{p|q}(\mathbb{C})$, one begins with a $\mathbb{Z}_2$-graded vector space $V = \mathbb{C}^p \oplus \mathbb{C}^q$ where one of the dimensions, say q, is even. Fixing a symmetric bilinear form s on $\mathbb{C}^p$ and an alternating bilinear form a on $\mathbb{C}^q$ (both of which are nondegenerate), one then takes the vector space $\mathfrak{osp}_{p|q}$ to be the set of complex linear transformations X of V which are skew with respect to $s + a$. In matrix form,

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} ,$$

where X_{11} is a complex $p \times p$ matrix which is skew w.r.t. the symmetric form s:

$$s(X_{11}v, w) + s(v, X_{11}w) = 0 \quad (v, w \in \mathbb{C}^p) ;$$

the complex $q \times q$ matrix X_{22} is skew w.r.t. the alternating form a; and so on.

In the case of $\mathfrak{g} = \mathfrak{gl}_{p|q}(\mathbb{C})$, the blocks X_{ij} are just complex matrices of the specified dimension, with no additional properties. In both cases ($\mathfrak{g} = \mathfrak{gl}, \mathfrak{osp}$) the $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is given by

$$X = \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} + \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} .$$

Let now G_0 be a complex Lie group with Lie algebra $\mathrm{Lie}(G_0) = \mathfrak{g}_0 \subset \mathfrak{g}$. In the case of $\mathfrak{g} = \mathfrak{gl}_{p|q}$ we take $G_0 = \mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C})$, i.e., the set of matrices

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} ,$$

with g_1 an invertible $p \times p$ matrix and g_2 an invertible $q \times q$ matrix. In the case of $\mathfrak{g} = \mathfrak{osp}_{p|q}$ we take $G_0 = \mathrm{SO}_p(\mathbb{C}) \times \mathrm{Sp}_q(\mathbb{C})$.

The corresponding Lie supergroup $G = \mathrm{GL}(p|q)$ or $G = \mathrm{OSp}(p|q)$ can in principle be constructed by invoking anticommuting parameters $\xi_1, \xi_2, \ldots$ to exponentiate also the odd part $\mathfrak{g}_1$ of the Lie superalgebra. However, in this approach (which is, roughly speaking, the one taken by Berezin) the difficult part is to explain exactly what is meant by the group multiplication law. The simplified approach to be outlined here is to describe the group multiplication only at the infinitesimal level, as follows.

With the aim of constructing a certain important vector bundle (see below), we view G_0 as a homogeneous space $\mathcal{M} := (G_0 \times G_0)/G_0$. In other words, G_0 is taken to act on $(g,h) \in G_0 \times G_0$ on the right by $(k,k) \in \mathrm{diag}(G_0 \times G_0) \cong G_0$, and we divide by that action, i.e., we form cosets $(g,h) \cdot G_0$ by the equivalence relation

$$(g,h) \sim (gk, hk) .$$

Yet another way of describing the situation is to say that we have a principal bundle $G_0 \times G_0 \to \mathcal{M}$ with structure group G_0.

Now G_0 also acts on the odd part $\mathfrak{g}_1$ of the Lie superalgebra $\mathfrak{g}$ by the adjoint representation, Ad. In the cases at hand this is the action

$$\mathrm{Ad} \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} : \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & g_1 X_{12} g_2^{-1} \\ g_2 X_{21} g_1^{-1} & 0 \end{pmatrix} .$$

It is then a basic fact of differential geometry that one can construct an *associated vector bundle* $E \to \mathcal{M}$ with total space

$$E = (G_0 \times G_0) \times_{G_0} \mathfrak{g}_1 .$$

The vectors of E over a point $(g,h) \cdot G_0$ of $\mathcal{M}$ are equivalence classes

$$[(gk,hk);Y] = [(g,h);\mathrm{Ad}(k)Y] \qquad (Y \in \mathfrak{g}_1,\, k \in G_0) .$$

Given the vector bundle $E \to \mathcal{M}$, consider the algebra of smooth sections of the exterior bundle of the dual of E:

$$\mathcal{A} := \Gamma(\mathcal{M}, \wedge E^*) .$$

$\wedge E^*$ is the vector bundle whose fibre over a point $p \in \mathcal{M}$ is the exterior algebra $\wedge E_p^*$ (with $E_p^* \cong \mathfrak{g}_1^*$, the dual vector space of $\mathfrak{g}_1$). Note that the adjoint representation Ad of G_0 on $\mathfrak{g}_1$ induces a representation $(\mathrm{Ad}^*)^{-1}$ of G_0 on $\mathfrak{g}_1^*$. Of course the exterior multiplication in the fibre over $(g,h) \cdot G_0 \in \mathcal{M}$ is defined by

$$[(g,h);A] \wedge [(g,h);B] = [(g,h);A \wedge B] \quad (A, B \in \wedge \mathfrak{g}_1^*) ,$$

which is independent of the choice of representative of the equivalence class.

The basic idea here is that a Lie group (like any manifold) can be described either by its points, or by the algebra of its functions. In the case of a Lie supergroup, the latter viewpoint is the good one, i.e., one uses a description by the algebra of (super)functions, and this algebra is none other than $\mathcal{A} = \Gamma(\mathcal{M}, \wedge E^*)$.

3.2. $\mathfrak{g}$-action on a Lie supergroup. — A useful fact about $\mathcal{A}$ is that its elements are in one-to-one correspondence with maps $F \in C^\infty(G_0 \times G_0, \wedge\, \mathfrak{g}_1^*)^{G_0}$, i.e., with smooth functions F on $G_0 \times G_0$ that take values in $\wedge\, \mathfrak{g}_1^*$ and are equivariant w.r.t. G_0:

$$F(g,h) = \mathrm{Ad}^*(k^{-1}) F(gk, hk) \; .$$

Indeed, such a function F determines a well-defined and unique section $s \in \mathcal{A}$ by

$$s\big((g,h)\cdot G_0\big) = [(g,h); F(g,h)] \; .$$

The algebra $\mathcal{A}$ has a natural $\mathbb{Z}_2$-grading $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ by

$$\mathcal{A}_0 = \bigoplus\nolimits_{\ell \text{ even}} \Gamma(\mathcal{M}, \wedge^\ell E^*) \; , \quad \mathcal{A}_1 = \bigoplus\nolimits_{\ell \text{ odd}} \Gamma(\mathcal{M}, \wedge^\ell E^*) \; .$$

The numerical (or $\mathbb{C}$-number) part of a section $s \in \mathcal{A}$ will be denoted by $\mathrm{num}(s)$. Note that $\mathrm{num}(s) \in \Gamma(\mathcal{M}, \wedge^0 E^*)$ vanishes for odd sections, $s \in \mathcal{A}_1$.

An *even derivation* of $\mathcal{A}$ is a first-order differential operator $D : \mathcal{A}_0 \to \mathcal{A}_0$ and $D : \mathcal{A}_1 \to \mathcal{A}_1$, which satisfies the Leibniz product rule. An *odd derivation* is a first-order differential operator $D : \mathcal{A}_0 \to \mathcal{A}_1$ and $D : \mathcal{A}_1 \to \mathcal{A}_0$ satisfying the anti-Leibniz rule.

We are now in a position to specify the Lie supergroup structure. The Lie group G_0 acts on $\mathcal{A} \cong C^\infty(G_0 \times G_0, \wedge \mathfrak{g}_1^*)^{G_0}$ in the obvious manner: there is a left G_0-action by

$$(L_g F)(h,h') = F(g^{-1}h, h') \; ,$$

and a right G_0-action by

$$(R_g F)(h,h') = F(h, g^{-1}h') \; ,$$

Specializing these to the infinitesimal level we get the corresponding canonical actions of the Lie algebra $\mathfrak{g}_0$ by even derivations of $\mathcal{A}$:

$$(\hat{X}^L F)(g,h) := \frac{d}{dt} F(\mathrm{e}^{-tX} g, h)\Big|_{t=0} \; , \quad (\hat{X}^R F)(g,h) := \frac{d}{dt} F(g, \mathrm{e}^{-tX} h)\Big|_{t=0} \; .$$

Note the representation property $[\hat{X}_1^L, \hat{X}_2^L] = \widehat{[X_1, X_2]}^L$ and $[\hat{X}_1^R, \hat{X}_2^R] = \widehat{[X_1, X_2]}^R$.

Similarly, the odd part $\mathfrak{g}_1$ of the Lie superalgebra $\mathfrak{g}$ acts in a canonical way by odd derivations of the algebra $\mathcal{A}$. This means that for every $Y \in \mathfrak{g}_1$ we are given (odd) first-order differential operators $\hat{Y}^L$ and $\hat{Y}^R$ such that the (super)bracket relations

$$[\hat{Y}_1^i, \hat{Y}_2^i] = \widehat{[Y_1, Y_2]}^i \; , \quad [\hat{X}^i, \hat{Y}^i] = \widehat{[X, Y]}^i$$

hold for $i = L, R$ and all $X \in \mathfrak{g}_0$ and $Y, Y_1, Y_2 \in \mathfrak{g}_1$. Unlike the action of the Lie algebra $\mathfrak{g}_0$, that of the odd part $\mathfrak{g}_1$ does not readily exponentiate (see, however, the work of Berezin [**B**]). We therefore leave this at the infinitesimal level.

We will not here go into a detailed exposition of the action of $\mathfrak{g}_1$ and all its properties. Suffice it to say that, e.g., the left action comes about by moving the left factor in

$$\mathrm{e}^{-\xi Y} g\, \mathrm{e}^{\sum_i \xi_i Y_i} h^{-1}$$

to the middle by $\mathrm{e}^{-\xi Y} g = g\, \mathrm{e}^{-\xi\,\mathrm{Ad}(g)^{-1} Y}$, using the Baker-Campbell-Hausdorff formula on the product

$$\mathrm{e}^{-\xi\,\mathrm{Ad}(g)^{-1} Y}\, \mathrm{e}^{\sum_i \xi_i Y_i} ,$$

and finally linearizing in ξ.

Let us now summarize the educational material of this subsection.

Definition 3.1. — *Given a complex Lie superalgebra $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}_1$, and a complex Lie group $G_0 = \exp(\mathfrak{g}_0)$ acting on the vector space $\mathfrak{g}_1$ by the adjoint representation* Ad, *there is an associated vector bundle $E \to \mathcal{M}$ with total space $E = (G_0 \times G_0) \times_{G_0} \mathfrak{g}_1$ over $\mathcal{M} = (G_0 \times G_0)/G_0$. By the Lie supergroup $G = (\mathfrak{g}, G_0)$ we mean the $\mathbb{Z}_2$-graded algebra of sections $\mathcal{A}_G = \Gamma(\mathcal{M}, \wedge E^*)$ carrying the canonical left and right action of $\mathfrak{g}$. A section of $\mathcal{A}_G$ is also referred to as a (super)function on G. The component in* $\mathrm{C}^\infty(\mathcal{M})$ *of a section $s \in \mathcal{A}_G$ is called the numerical part of s and is denoted by* $\mathrm{num}(s)$.

3.3. What's a representation in the supergroup setting?— If G is a Lie group (or, for that matter, any group), a representation of G is given by a complex vector space V and a homomorphism from G into the group of linear transformations of V, i.e., a mapping $\rho : G \to \mathrm{GL}(V)$ which respects the group multiplication law:

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2) .$$

In the Lie supergroup setting, since we have purposely avoided defining what is meant by the group multiplication law, we have to give meaning to the word "representation" in an alternative (albeit equivalent) way.

Recall that the basic data of a complex Lie supergroup G are a complex Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and a complex Lie group G_0 with $\mathrm{Lie}(G_0) = \mathfrak{g}_0$. To specify a representation of G, one first of all needs a representation space, which in the present context has to be a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$. For present use recall that

$$\mathrm{STr}_V = \mathrm{Tr}_{V_0} - \mathrm{Tr}_{V_1} .$$

The $\mathbb{Z}_2$-grading of V induces a natural $\mathbb{Z}_2$-grading of the endomorphisms of V:

$$\mathrm{End}(V) = \mathrm{End}_0(V) \oplus \mathrm{End}_1(V) .$$

An element $A \in \mathrm{End}_0(V)$ is a linear mapping $A : V_0 \to V_0$ and $A : V_1 \to V_1$, while an element $B \in \mathrm{End}_1(V)$ is a linear mapping $B : V_0 \to V_1$ and $B : V_1 \to V_0$. By this $\mathbb{Z}_2$-grading the algebra $\mathrm{End}(V)$ carries the natural structure of a Lie superalgebra.

Definition 3.2. — *A representation of a complex Lie supergroup $G = (\mathfrak{g}, G_0)$ on a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ is given by a homomorphism of Lie superalgebras*

$$\rho_* : \mathfrak{g} \to \mathrm{End}(V) ,$$

and a homomorphism of Lie groups

$$\rho : G_0 \to \mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \;,$$

which are compatible in the sense that $(d\rho)_{\mathrm{Id}} = \rho_*|_{\mathfrak{g}_0}$.

Remark. — Note that compatibility implies that

$$\rho(g)\,\rho_*(Y)\,\rho(g^{-1}) = \rho_*\big(\mathrm{Ad}(g)Y\big) \quad (g \in G_0\,, Y \in \mathfrak{g}_1) \;.$$

3.4. What's a character of G?— If we are given a representation (V,ρ_*,ρ) of the Lie supergroup $G = (\mathfrak{g}, G_0)$, we can say precisely – in the language of Sect. 3.1 – what is meant by its character χ, which we write informally as $\chi(M) = \mathrm{STr}_V\,\rho(M)$. The precise meaning of χ comes from its definition as an element of $\mathcal{A}_G = \Gamma(\mathcal{M}, \wedge E^*)$, i.e., as a section of the vector bundle

$$\wedge E^* = (G_0 \times G_0) \times_{G_0} \wedge \mathfrak{g}_1^* \;\to\; \mathcal{M} = (G_0 \times G_0)/G_0 \;.$$

This section χ is constructed from the representation (V,ρ_*,ρ) as follows.

Fix some basis $Y_1, Y_2, \ldots$ of $\mathfrak{g}_1$, and denote the corresponding dual basis by $\xi_1, \xi_2, \ldots$. (The construction to be made does not depend on which basis is chosen.) Then let $\xi \in \mathrm{End}(\mathfrak{g}_1) \cong \mathfrak{g}_1 \otimes \mathfrak{g}_1^*$ be the tautological map, i.e.,

$$\xi = \textstyle\sum_i Y_i \otimes \xi_i$$

with the following interpretation. The $Y_i \in \mathfrak{g}_1$, as well as their representations $\rho_*(Y_i) \in \mathrm{End}_1(V)$, are viewed as matrices and their destiny is *to be multiplied as matrices*. The ξ_i, on the other hand, are regarded as coordinates or linear functions on $\mathfrak{g}_1$; as such they are to be multiplied in the *graded-commutative* sense. (Since $\mathfrak{g}_1$ is the *odd* part of a Lie superalgebra, consistency requires that the product among the ξ_i be the exterior one). Thus we may also view the ξ_i as a set of generators of the exterior algebra $\wedge\mathfrak{g}_1^*$.

Now, fixing some pair $(g,h) \in G_0 \times G_0$, consider the task of representing $M = g\,\mathrm{e}^{\xi}h^{-1}$ as an operator on V, and form the supertrace of this operator to define

$$\begin{aligned} \chi(g,h) \;&:=\; \mathrm{STr}_V\,\rho(g)\,\mathrm{e}^{\sum_i \xi_i \rho_*(Y_i)}\rho(h^{-1}) \\ &=\; \mathrm{STr}_V\,\rho(gh^{-1}) + \textstyle\sum_i \xi_i\,\mathrm{STr}_V\,\rho(g)\rho_*(Y_i)\rho(h^{-1}) \\ &+\; \tfrac{1}{2}\textstyle\sum_{i,j} \xi_i\xi_j\,\mathrm{STr}_V\,\rho(g)\rho_*(Y_i)\rho_*(Y_j)\rho(h^{-1}) + \ldots \;, \end{aligned}$$

where the Taylor expansion of the exponential function terminates at finite order since the ξ_i are nilpotent. If the dimension of V is finite, this definition makes perfect sense. In the infinite-dimensional case, convergence issues arise and $\chi(g,h)$ will typically exist only for (g,h) in some open domain $\mathcal{U} \subset G_0 \times G_0$.

When g and h are allowed to vary, χ becomes a function on $\mathcal{U}$ which takes values in $\wedge g_1^*$ and is equivariant with respect to G_0. Indeed,

$$\chi(gk,hk) = \mathrm{STr}_V\,\rho(g)\,\mathrm{e}^{\sum_i \xi_i\,\rho(k)\rho_*(Y_i)\rho(k^{-1})}\rho(h^{-1}) \;.$$

By the compatibility of ρ and ρ_*,

$$\sum_i \xi_i\, \rho(k)\rho_*(Y_i)\rho(k^{-1}) = \sum_{i,j} \xi_i\, \rho_*\big(Y_j \operatorname{Ad}(k)_{ji}\big) = \sum_{i,j} \xi_i \operatorname{Ad}^*(k)_{ij}\, \rho_*(Y_j)\ ,$$

and therefore

$$\chi(g,h) = \operatorname{Ad}^*(k^{-1})\, \chi(gk, hk)\ ,$$

where $k \mapsto \operatorname{Ad}^*(k^{-1})$ is the induced G_0-representation on $\wedge g_1^*$. Hence χ is an element of $\mathrm{C}^\infty(\mathcal{U}, \wedge\mathfrak{g}_1^*)^{G_0}$, or equivalently a section of $\mathcal{A} = \Gamma(\mathcal{U}/G_0, \wedge E^*)$.

Definition 3.3. — *The character determined by the representation* (V, ρ_*, ρ) *of a Lie supergroup* $G = (\mathfrak{g}, G_0)$ *is defined to be the section* $\chi \in \mathcal{A}_G$ *given by*

$$\chi(g,h) = \operatorname{STr}_V \rho(g)\, \mathrm{e}^{\sum_i \xi_i\, \rho_*(Y_i)} \rho(h^{-1})\ ,$$

whenever this exists. We also write $M = g\,\mathrm{e}^{\xi} h^{-1}$ *and* $\chi(M) = \operatorname{STr}_V \rho(M)$ *for short.*

Remark. — If we identify the $\mathbb{C}$-valued component $\Gamma(\mathcal{M}, \wedge^0 E^*) \cong \mathrm{C}^\infty(G_0 \times G_0, \mathbb{C})^{G_0}$ with $\mathrm{C}^\infty(G_0)$ by $F(g,h) = f(gh^{-1})$, the numerical part of χ,

$$\operatorname{num}(\chi)(g,h) = \big(\operatorname{Tr}_{V_0} - \operatorname{Tr}_{V_1}\big)\, \rho(gh^{-1})\ ,$$

coincides with the (super)character of the complex Lie group G_0 associated with the $\mathbb{Z}_2$-graded G_0-representation $(V_0 \oplus V_1, \rho)$.

3.5. χ is an eigenfunction of all Laplace-Casimir operators. — Irreducible characters are special functions with special properties. Foremost among these is their being joint eigenfunctions of the ring of invariant differential operators. Let us review this general property in the superalgebra setting, which is where it will be exploited below.

If $\mathfrak{g}$ is a Lie superalgebra, a *Casimir invariant* of $\mathfrak{g}$ is an element I in the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$, i.e, a polynomial I in the generators of $\mathfrak{g}$ with the property $[I, X] = 0$ for all $X \in \mathfrak{g}$. For example, if $E_1, \ldots, E_d$ (with $d = \dim \mathfrak{g}$) is a basis of $\mathfrak{g}$ and $Q : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a nondegenerate invariant quadratic form, the quadratic Casimir invariant is

$$I_2 = \sum_{ij} Q^{ij} E_i E_j\ ,$$

where the coefficients Q^{ij} are determined by $\sum_j Q^{ij} Q_{jk} = \delta^i_k$ and $Q_{ij} = Q(E_i, E_j)$.

***Lemma 3.4* (Berezin)**. — *Denote the canonical generators of the Lie superalgebra* $\mathfrak{gl}_{p|q}$ *by* $E^{\sigma\tau}_{ij}$ *where the range of a lower index is understood to be* $i \in \{1, 2, \ldots, p\}$ *if the corresponding upper index is* $\sigma = 1$, *and* $i \in \{1, 2, \ldots, q\}$ *if* $\sigma = 2$. *The homogeneous Casimir invariant of degree* $\ell \in \mathbb{N}$ *of* $\mathfrak{gl}_{p|q}$ *then is expressed as*

$$I_\ell = \sum E^{\sigma_1\sigma_2}_{j_1 j_2} (-1)^{\sigma_2} E^{\sigma_2\sigma_3}_{j_2 j_3} (-1)^{\sigma_3} \cdots E^{\sigma_{\ell-1}\sigma_\ell}_{j_{\ell-1} j_\ell} (-1)^{\sigma_\ell} E^{\sigma_\ell \sigma_1}_{j_\ell j_1}\ .$$

Remark. — From the basic bracket relations (3) it is in fact easy to check that $[X, I_\ell] = 0$ for all $X \in \mathfrak{gl}_{p|q}$.

Now turn to the functions F on a Lie supergroup $G = (\mathfrak{g}, G_0)$, i.e., to the sections F of the algebra $\mathcal{A}_G = \Gamma(\mathcal{M}, \wedge E^*)$. Every Casimir invariant $I \in \mathsf{U}(\mathfrak{g})$ determines an invariant differential operator – also called a Laplace-Casimir operator – $D(I)$ on such functions. In the case of the quadratic Casimir invariant I_2 this is

$$\big(D(I_2)F\big)(M) = \sum\nolimits_{ij} Q^{ij}\hat{E}_i^L \hat{E}_j^L F(M) = \sum\nolimits_{ij} Q^{ij}\hat{E}_i^R \hat{E}_j^R F(M) \,,$$

where $E_i \mapsto \hat{E}_i^{L,R}$ are the canonical left and right actions of the Lie superalgebra $\mathfrak{g}$ on the Lie supergroup G (Def. 3.1). More generally, $D(I)$ is obtained by replacing in the polynomial expression for I each generator of $\mathfrak{g}$ by the corresponding right-invariant or left-invariant differential operator.

Assume now an irreducible representation ρ of the Lie supergroup G to be given. (For notational simplicity we suppress the representation space V, and we no longer distinguish between ρ and the representation ρ_* of the Lie superalgebra $\mathfrak{g}$.) Applying any Laplace-Casimir operator $D(I)$ to the character $\chi(M) = \mathrm{STr}\,\rho(M)$ you get

$$\big(D(I)\chi\big)(M) = \mathrm{STr}\,\rho(I)\rho(M) \,.$$

Since the representation is irreducible, and I commutes with all the generators of $\mathfrak{g}$, the operator $\rho(I)$ is a multiple of the unit operator by Schur's lemma: $\rho(I) = \lambda(I) \times \mathrm{Id}$ with $\lambda(I) \in \mathbb{C}$. Thus χ is an eigenfunction:

$$D(I)\chi = \lambda(I)\,\chi \,.$$

This fact will be exploited in Sect. 4.3.

3.6. Radial functions. — Another important fact about the characters of a group G is that they are constant on conjugacy classes: $\chi(g) = \chi(hgh^{-1})$. In the case of a Lie group G, functions $F : G \to \mathbb{C}$ with this property are called *radial*. The infinitesimal version of the radial property is

$$0 = \frac{d}{ds} F(\mathrm{e}^{-sX} g\, \mathrm{e}^{sX})\Big|_{s=0} = \big(\hat{X}^L + \hat{X}^R\big)\,F(g) \,,$$

in which form the definition readily carries over to the Lie supergroup setting. Notice that the notation used here is consistent with our earlier definition of the left and right actions $\hat{X}^{L,R}$, by the identification $F_2(g,h) = F_1(gh^{-1})$.

Definition 3.5. — *A section $F \in \mathcal{A}_G$ on a Lie supergroup G is called radial if*

$$\big(\hat{X}^L + \hat{X}^R\big)\,F = 0$$

holds for all X in the Lie superalgebra of G.

Now let G be a connected compact Lie group. The set of conjugacy classes then is parameterized by the elements t in a Cartan subgroup, or maximal torus, T. Thus a radial function F on G determines a function f on T by restriction, $f := F\big|_T$. Conversely, a function f on T extends to a radial function F on G provided that f is invariant under the action of the Weyl group W of G, i.e. $f(t) = f(w \cdot t)$ for all $w \in W$.

Turning to the case of a Lie supergroup G, consider the situation where the underlying Lie group G_0 is compact (and connected), with maximal torus T. A radial section $F \in \mathcal{A}_G$ still determines a function $f : T \to \mathbb{C}$ by restriction and truncation to the numerical part,

$$f = \pi(F) := \mathrm{num}(F)\big|_T ,$$

and this function f is still invariant under the Weyl group W of G_0. However, the converse is no longer true; in order for a W-invariant function f on T to extend to a radial section $F \in \mathcal{A}_G$ some extra conditions must be fulfilled.

To specify what these are, let Δ_1 denote the set of odd roots of the Lie superalgebra $\mathfrak{g}$, i.e., the set of functions $\beta : \mathfrak{t} = \mathrm{Lie}(T) \to \mathbb{C}$ with the property

$$\forall H \in \mathfrak{t} : \quad [H, X_\beta] = \beta(H) X_\beta \quad (X_\beta \in \mathfrak{g}_1) .$$

If $\langle \cdot , \cdot \rangle$ is a nondegenerate and invariant quadratic form on $\mathfrak{g}$, assign to each $\beta \in \Delta_1$ its dual element $H_\beta \in \mathfrak{t}$ by $\langle H_\beta , \cdot \rangle = \beta$.

***Theorem 3.6* (Berezin).** — *Let $f : T \to \mathbb{C}$ be a W-invariant function on the maximal torus T of a Lie supergroup G. Then f extends to a radial function $F \in \mathcal{A}_G$ if and only if the following condition is satisfied for every $\beta \in \Delta_1$: the limit*

$$\lim_{H \to H'} \beta(H)^{-1} \frac{d}{ds} f(\mathrm{e}^{H + sH_\beta})\Big|_{s=0}$$

exists for all H' in the zero locus of β, i.e., on the set $\{H' \in \mathfrak{t} \,|\, \beta(H') = 0\}$.

***Remark*.** — The necessity of this condition for extendability can be seen as follows. Let $F \in \mathcal{A}_G$ be a radial section, and put $f = \pi(F)$. An element $H_\beta \in \mathfrak{t}$ acts on f by

$$(H_\beta \cdot f)(\mathrm{e}^H) = \frac{d}{ds} f(\mathrm{e}^H \mathrm{e}^{sH_\beta})\Big|_{s=0} .$$

If $\hat{H}_\beta^{L,R}$ are the differential operators associated to H_β by the left and right actions of $\mathfrak{g}$ on $\mathcal{A}_G$, we equivalently have

$$H_\beta \cdot f = \pi\big(\hat{H}_\beta^R F\big) = -\pi\big(\hat{H}_\beta^L F\big) .$$

Now recall $\langle H, H_\beta \rangle = \beta(H)$ and notice

$$\big\langle H, [X_\beta, X_{-\beta}] \big\rangle = \big\langle [H, X_\beta], X_{-\beta} \big\rangle = \beta(H) \, \big\langle X_\beta, X_{-\beta} \big\rangle .$$

Hence if root vectors are normalized so that $\langle X_\beta, X_{-\beta} \rangle = 1$, then $[X_\beta, X_{-\beta}] = H_\beta$. By the representation property this carries over to a relation between differential operators:

$$\hat{H}_\beta^i = [\hat{X}_\beta^i, \hat{X}^i_{-\beta}] \quad (i = L, R) .$$

Therefore, the directional derivative $H_\beta \cdot f$ can be expressed as

$$\begin{aligned} H_\beta \cdot f &= \pi\big(\tfrac{1}{2}(\hat{H}^R_\beta - \hat{H}^L_\beta)F\big) \\ &= \tfrac{1}{2}\pi\big([\hat{X}^R_\beta, \hat{X}^R_{-\beta}]F - [\hat{X}^L_\beta, \hat{X}^L_{-\beta}]F\big) \\ &= \tfrac{1}{2}\pi\big([\hat{X}^L_\beta + \hat{X}^R_\beta, \hat{X}^R_{-\beta} - \hat{X}^L_{-\beta}]F\big) \ , \end{aligned}$$

where the last equality sign follows from the fact that the left and right $\mathfrak{g}$-actions on $\mathcal{A}_G$ commute in the graded-commutative sense.

Next we use the property that the section F is radial: $(\hat{X}^L_\beta + \hat{X}^R_\beta)F = 0$. Abbreviating $F' := \frac{1}{2}(\hat{X}^R_{-\beta} - \hat{X}^L_{-\beta})F$ we then arrive at an expression for $H_\beta \cdot f$ as

$$H_\beta \cdot f = \pi\big((\hat{X}^L_\beta + \hat{X}^R_\beta)F'\big) \ .$$

Since the differential operator $\hat{X}^L_\beta + \hat{X}^R_\beta$ represents the adjoint action

$$-\mathrm{ad}(X_\beta) : H \mapsto [H, X_\beta] = \beta(H) X_\beta \ ,$$

it is clear that $H_\beta \cdot f$ vanishes linearly on the zero locus of β and $\beta^{-1} H_\beta \cdot f$ exists.

Thus we have explained why the condition stated in Theorem 3.6 is a necessary one. For sufficiency we refer to Berezin [**B**].

3.7. Radial part of the Laplacians for $\mathrm{U}(p|q)$**.—** Recall that for every Casimir invariant $I \in \mathsf{U}(\mathfrak{g})$ there is an invariant differential operator $D(I)$ on the algebra of sections $\mathcal{A}_G$. If $F \in \mathcal{A}_G$ is radial, i.e., $(\hat{X}^L + \hat{X}^R)F = 0$ for all $X \in \mathfrak{g}$, then so is $D(I)F$, since $D(I)$ commutes with all of the $\hat{X}^{L,R}$. Therefore, every invariant differential operator $D(I)$ has a radial part, $\dot{D}(I)$; if π is the restriction from the radial sections F in $\mathcal{A}_G$ to the torus functions $f : T \to \mathbb{C}$, and π^{-1} is the inverse map on the extendable W-invariant torus functions, this is the differential operator $\dot{D}(I) = \pi \circ D(I) \circ \pi^{-1}$.

We finish our exposition of background material by writing down the radial parts of the invariant differential operators for the unitary Lie supergroup $G = \mathrm{U}(p|q)$, whose base manifold is the compact real Lie group $G_0 = \mathrm{U}(p) \times \mathrm{U}(q)$. The Casimir invariants I_ℓ of $\mathfrak{gl}_{p|q}$ were given in Lemma 3.4.

Let the maximal torus $T = \mathrm{U}(1)^{p+q}$ of $\mathrm{U}(p) \times \mathrm{U}(q)$ be coordinatized by

$$t = \mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\psi_1}, \ldots, \mathrm{e}^{\mathrm{i}\psi_p}; \mathrm{e}^{\mathrm{i}\varphi_1}, \ldots, \mathrm{e}^{\mathrm{i}\varphi_q}\right) \ ,$$

and denote by $\tilde{D}_\ell$ the homogeneous radial differential operators

$$\tilde{D}_\ell = \sum_{i=1}^{p} \frac{\partial^\ell}{\partial \psi_i^\ell} - (-1)^\ell \sum_{j=1}^{q} \frac{\partial^\ell}{\partial \varphi_j^\ell} \ .$$

***Theorem 3.7* (Berezin)**. — *If $J : \mathrm{U}(1)^{p+q} \to \mathbb{R}$ is the function given by*

$$J = \frac{\prod_{1 \le i < i' \le p} \sin^2\left(\frac{1}{2}(\psi_i - \psi_{i'})\right) \prod_{1 \le j < j' \le q} \sin^2\left(\frac{1}{2}(\varphi_j - \varphi_{j'})\right)}{\prod_{i=1}^{p} \prod_{j=1}^{q} \sin^2\left(\frac{1}{2}(\psi_i - \varphi_j)\right)} \ ,$$

the degree-ℓ Laplace-Casimir operator for $\mathrm{U}(p|q)$ *has the radial part*

$$\dot{D}(I_\ell) = J^{-1/2}\left(\tilde{D}_\ell + P_{\ell-1}\right) \circ J^{1/2} \;,$$

where $P_{\ell-1}$ *denotes lower-order terms, which are polynomial of maximal degree* $\ell - 1$ *in the homogeneous operators* $\tilde{D}_k$.

In the special case when dimensions match, one has the following simplification, which will be of use in Sect. 4.5.

Lemma 3.8. — *For* $p = q = n$ *the function* J *can be put in the form* $J = \mathrm{Det}^2(K)$ *where* K *is the* $n \times n$ *matrix with entries*

$$K_{ij} = \frac{1}{\sin\left(\frac{1}{2}(\psi_i - \varphi_j)\right)} \;.$$

Proof. — In the Cauchy determinant formula

$$\frac{\prod_{i<j}(x_i - x_j)(y_j - y_i)}{\prod_{i,j}(x_i - y_j)} = \mathrm{Det}\left(\frac{1}{x_i - y_j}\right)_{i,j=1,\ldots,n} \;,$$

make the substitution $x_i = \mathrm{e}^{\mathrm{i}\psi_i}$ and $y_j = \mathrm{e}^{\mathrm{i}\varphi_j}$. The statement then immediately follows from Euler's formula $2\mathrm{i}\sin z = \mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}$ on dividing both sides by a suitable factor. □

Corollary 3.9. — *For* $p = q$ *the constant term of the lower-order differential operator* $P_{\ell-1}(\partial/\partial\psi_i, \partial/\partial\varphi_j)$ *in the radial part* $\dot{D}(I_\ell)$ *vanishes for all* $\ell \in \mathbb{N}$*:* $P_{\ell-1}(0,0) = 0$.

Proof. — The first step is to show that for $p = q = n$ the square root $J^{1/2}$ is annihilated by every homogeneous radial differential operator $\tilde{D}_\ell$. For that purpose, write

$$J^{1/2} = \mathrm{Det}(K) = (2\mathrm{i})^n \mathrm{e}^{-\frac{\mathrm{i}}{2}\sum_k(\psi_k - \varphi_k)}\mathrm{Det}\left(\frac{1}{1 - \mathrm{e}^{-\mathrm{i}(\psi_i - \varphi_j)}}\right)_{i,j=1,\ldots,n} \;.$$

Now express the determinant as a sum over permutations, move the ψ_i and φ_j into the lower resp. upper half of the complex plane, and expand the factors $\left(1 - \mathrm{e}^{-\mathrm{i}(\psi_i - \varphi_j)}\right)^{-1}$ into geometric series. The result is an absolutely convergent series, each term of which is annihilated by every one of the $\tilde{D}_\ell$. Thus $\tilde{D}_\ell J^{1/2} = 0$ for all $\ell \in \mathbb{N}$.

Since $\dot{D}(I_\ell)$ is a differential operator, one has $\dot{D}(I_\ell) \cdot 1 = 0$ and Theorem 3.7 implies $0 = J^{-1/2}(\tilde{D}_\ell + P_{\ell-1})J^{1/2} = P_{\ell-1}(0,0)$. □

Finding the radial part of an invariant differential operator is an algebraic problem (as opposed to an analytical one). It is therefore clear that the formulas given in Theorem 3.7 hold not only for the compact Lie supergroup $\mathrm{U}(p|q)$ but admit analytic continuation to an open domain in the complex Lie supergroup $\mathrm{GL}(p|q)$. It is in such a domain that we will use them to complete the proof of Theorem 1.1.

4. Computation of the $\mathrm{GL}(2n|2n)$-character χ

In Sect. 2.4 we considered the Howe pair $\mathfrak{gl}_{2n|2n} \times \mathrm{U}(N)$ acting on the oscillator-spinor module $\mathcal{V} = \wedge(\mathbb{C}^{2n} \otimes \mathbb{C}^N) \otimes S(\mathbb{C}^{2n} \otimes \mathbb{C}^N)$, and showed the correlation function of interest to be an irreducible character, χ; in fact, combining Props. 2.2 and 2.4 with Cor. 2.3 we have that the left-hand side of the statement of Theorem 1.1 is equal to

$$\int_{\mathrm{U}(N)} \mathrm{SDet}^{-1}(\mathrm{Id} - t \otimes U)\, dU = \mathrm{STr}_{\mathcal{V}_0}\, \tilde{\rho}(t) = \chi(t) \;,$$

where $\mathcal{V}_0$ is the vector space of $\mathrm{U}(N)$-invariants in $\mathcal{V}$, and $\tilde{\rho}$ is the $\mathfrak{gl}_{2n|2n}$-representation arising from the standard one by changing the normal ordering (Sect. 2.3). In the present section we adopt the simplified notations $V := \mathcal{V}_0$ and $\rho := \tilde{\rho}$.

4.1. Extending the domain of definition of χ. —

All of our considerations of χ in Sect. 2 were restricted to the diagonal matrices $M = t$; it is, in fact, only the diagonal values $\chi(t)$ that we ultimately want to know. However, to establish good analytical control and actually compute these values, we will now extend the torus function $t \mapsto \chi(t)$ to a function on the Lie supergroup $G = \mathrm{GL}(2n|2n)$.

Because the representation space V is infinite-dimensional, the character χ at hand does not extend to all of the complex Lie group $G_0 = \mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C})$, but exists only on a suitable open domain in G_0. Our first task is to describe this domain.

Proposition 4.1. — *The character $\chi(t) = \mathrm{STr}_V\, \rho(t)$ extends to an analytic function on the product manifold $\mathcal{U} = \mathcal{U}^{(1)} \times \mathcal{U}^{(2)} \subset G_0$ where $\mathcal{U}^{(1)} = \mathrm{U}(2n)$ and*

$$\mathcal{U}^{(2)} = \{T \,\mathrm{diag}(C,D)\, T^{-1} \mid C,D \in \mathrm{Herm}(\mathbb{C}^n);\, 0 < C < 1 < D;\, T \in \mathrm{U}(n,n)\} \,.$$

This analytic function $\chi : \mathcal{U} \to \mathbb{C}$ analytically continues to a holomorphic function on a tubular complex neighborhood $\tilde{\mathcal{U}}$ of $\mathcal{U}$ in G_0. The holomorphic function $\chi : \tilde{\mathcal{U}} \to \mathbb{C}$ in turn lifts to a holomorphic even section of $\Gamma(\tilde{\mathcal{U}}, \wedge E^*)$.

Proof. — From Sects. 2.3 and 2.4 we know that $\mathrm{STr}_V\, \rho(t)$ has the expression

$$\mathrm{STr}_V\, \rho\big(\mathrm{diag}(t_1,t_2)\big) = \int_{\mathrm{U}(N)} \frac{\mathrm{Det}(\mathrm{Id} - t_1 \otimes U)}{\mathrm{Det}(\mathrm{Id} - t_2 \otimes U)}\, dU \qquad (7)$$

and, parameterizing the diagonal matrices $t = \mathrm{diag}(t_1,t_2)$ as

$$t_1 = \mathrm{diag}(\mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_n}, \mathrm{e}^{\beta_1}, \ldots, \mathrm{e}^{\beta_n}) \;, \quad t_2 = \mathrm{diag}(\mathrm{e}^{\gamma_1}, \ldots, \mathrm{e}^{\gamma_n}, \mathrm{e}^{\delta_1}, \ldots, \mathrm{e}^{\delta_n}) \;, \qquad (8)$$

exists for $\mathfrak{Re}\,\gamma_j < 0 < \mathfrak{Re}\,\delta_j$ and arbitrary complex α_j and β_j ($j = 1, \ldots, n$).

Consider now any $M = \mathrm{diag}(M_1, M_2) \in \mathcal{U}^{(1)} \times \mathcal{U}^{(2)}$. Every such matrix M can be diagonalized: $M = gtg^{-1}$ where $g \in \mathrm{U}(2n) \times \mathrm{U}(n,n)$, and $t = \mathrm{diag}(t_1,t_2)$ is given by (8) with parameters in the good range: $\gamma_j < 0 < \delta_j$ and $\alpha_j, \beta_j \in \mathrm{i}\mathbb{R}$.

Since the representation ρ is unitary for $\mathrm{U}(2n) \times \mathrm{U}(n,n)$ (see Sect. 2.3), we have $\rho(g^{-1}) = \rho(g)^{-1} = \rho(g)^*$ and hence

$$\chi(M) = \mathrm{STr}_V\, \rho(g)\rho(t)\rho(g)^{-1} = \chi(t) .$$

Thus $\chi(M)$ exists on $\mathcal{U}$ and is constant on the orbits of the adjoint action of $\mathrm{U}(2n) \times \mathrm{U}(n,n)$ on $\mathcal{U}$. The same holds true for the right-hand side of (7). This relation therefore extends to

$$\chi(M) = \mathrm{STr}_V\, \rho(M) = \int_{\mathrm{U}(N)} \mathrm{SDet}^{-1}(\mathrm{Id} - M \otimes U) dU ,$$

where $M = \mathrm{diag}(M_1, M_2) \in \mathcal{U}$ is regarded as a block-diagonal supermatrix.

Since the function $M \mapsto \mathrm{SDet}^{-1}(\mathrm{Id} - M \otimes U)$ is analytic on $\mathcal{U}$ for every fixed unitary matrix U, so is its integral over the compact Lie group $\mathrm{U}(N)$. Thus $\chi : \mathcal{U} \to \mathbb{C}$ is an analytic function. Moreover, for all M in the interior of $\mathcal{U}$, i.e., for eigenvalues e^{γ_j} and e^{δ_j} that are strictly smaller resp. greater than unity, this analytic function has a finite radius of convergence. It therefore extends holomorphically to a tubular complex neighborhood $\tilde{\mathcal{U}}$ of $\mathcal{U}$. (Of course this neighborhood degenerates on approaching the boundary of $\mathcal{U}$ or sending any of the γ_j or δ_j to zero.)

To elevate $\chi : \tilde{\mathcal{U}} \to \mathbb{C}$ to a holomorphic section of $\Gamma(\tilde{\mathcal{U}}, \wedge E^*)$, recall the definitions $G_0 = \mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C})$ and $E = (G_0 \times G_0) \times_{G_0} \mathfrak{g}_1$ where $\mathfrak{g}_1$ is the odd part of the Lie superalgebra $\mathfrak{gl}_{2n|2n}$. The domain $\tilde{\mathcal{U}}$ is open in the complex Lie group G_0. If $M \in \tilde{\mathcal{U}}$, then for any pair $(g,h) \in G_0 \times G_0$ in the fibre over $M = gh^{-1}$ of the principal G_0-bundle $G_0 \times G_0 \to G_0$ we set

$$\chi(g,h) := \mathrm{STr}_V\, \rho(g)\, \mathrm{e}^{\sum_i \xi_i \rho_*(Y_i)} \rho(h^{-1}) .$$

This expression exists because the Taylor expansion with respect to the nilpotent generators $\xi_i \in \mathfrak{g}_1^*$ terminates at finite order and each expansion coefficient

$$\mathrm{STr}_V\, \rho(g)\rho_*(Y_{i_1})\rho_*(Y_{i_2}) \cdots \rho_*(Y_{i_k})\rho(h^{-1})$$

exists. Since the operators $\rho_*(Y_i)$ for $Y_i \in \mathfrak{g}_1$ lie in the odd subspace $\mathrm{End}_1(V)$, these expansion coefficients are nonzero only when the order k is even. Their dependence on g,h is holomorphic. Finally, by setting

$$\chi\big((g,h) \cdot G_0\big) := [(g,h); \chi(g,h)]$$

as usual, χ is established as a holomorphic even section of $\Gamma(\tilde{\mathcal{U}}, \wedge E^*)$. □

In the following we will frequently use the short-hand notation $\chi(M) = \mathrm{STr}_V\, \rho(M)$ for the section $\chi \in \Gamma(\mathcal{U}, \wedge E^*)$ which is obtained by restricting the holomorphic section $\chi \in \Gamma(\tilde{\mathcal{U}}, \wedge E^*)$ to the real-analytic domain $\mathcal{U}$ defined in Prop. 4.1.

Having extended the torus function $t \mapsto \chi(t)$ to a section on a domain of the Lie supergroup $\mathrm{GL}(2n|2n)$, we can now apply the powerful machinery reviewed in Sect. 3 and exploit the fact that χ is an eigenfunction of the ring of $\mathfrak{gl}_{2n|2n}$-invariant differential operators: $D(I)\chi = \lambda(I)\chi$. The first step is to calculate the eigenvalues $\lambda(I)$ from the relation $\rho(I) = \lambda(I) \times \mathrm{Id}_V$.

4.2. All Casimir invariants vanish on V. — From Lemma 3.4 recall the expression for the degree-ℓ Casimir invariant I_ℓ of $\mathfrak{gl}_{2n|2n}$. There exists a simple heuristic why all of these invariants must be identically zero in such a representation as (V,ρ).

The first step of the argument is to verify from the basic bracket relations (3) of $\mathfrak{gl}_{2n|2n}$ that each invariant I_ℓ can be expressed as an anticommutator of odd elements:

$$I_\ell = [Q, F^{(\ell)}] ,$$

where $Q = \sum_{i=1}^{2n} E_{ii}^{12}$, and $F^{(\ell)} \in \mathsf{U}(\mathfrak{gl}_{2n|2n})$ is given by

$$F^{(\ell)} = \sum E_{ij_2}^{2\sigma_2}(-1)^{\sigma_2} E_{j_2 j_3}^{\sigma_2\sigma_3}(-1)^{\sigma_3} \cdots E_{j_{\ell-1}j_\ell}^{\sigma_{\ell-1}\sigma_\ell}(-1)^{\sigma_\ell} E_{j_\ell i}^{\sigma_\ell 1} .$$

If the representation space V were finite-dimensional, we could now argue that

$$\mathrm{STr}_V\, \rho(I_\ell) = \mathrm{STr}_V\, [\rho(Q), \rho(F^{(\ell)})] = 0 ,$$

since the supertrace of any bracket vanishes. On the other hand, since I_ℓ is a Casimir invariant, the operator $\rho(I_\ell)$ on the irreducible representation space V must be a multiple of unity: $\rho(I_\ell) = \lambda(I_\ell) \times \mathrm{Id}_V$. In finite dimension we could therefore say that

$$\mathrm{STr}_V\, \rho(I_\ell) = \lambda(I_\ell)\, \mathrm{STr}_V\, \mathrm{Id} = (\dim V_0 - \dim V_1)\, \lambda(I_\ell) .$$

Inspection of V shows that there is exactly one vector in the even subspace V_0 which has no partner in the odd subspace V_1 – this vector is the "vacuum". Thus the dimension of V_0 exceeds that of V_1 by one. Hence $\mathrm{STr}_V\, \rho(I_\ell) = 1 \cdot \lambda(I_\ell)$, and from the previous result $\mathrm{STr}_V\, \rho(I_\ell) = 0$ we would be forced to conclude $\lambda(I_\ell) = 0$.

This argument is not sound, as the representation space V has infinite dimension and the traces $\mathrm{STr}_V\, \rho(I_\ell)$ and $\mathrm{STr}_V \mathrm{Id}$ as such do not exist. Nevertheless, the conclusion still holds true:

Proposition 4.2. — *In the representation (V,ρ) all Casimir invariants I_ℓ vanish.*

Proof. — The heuristic argument becomes rigorous on regularizing the traces. This is done with the help of an alternating sum of generators,

$$\Lambda = \sum_{\sigma=1}^{2} \sum_{j=1}^{n} (E_{jj}^{\sigma\sigma} - E_{j+n,j+n}^{\sigma\sigma}) ,$$

which is not a Casimir invariant of $\mathfrak{gl}_{2n|2n}$, but does lie in the center of $\mathsf{U}(\mathfrak{gl}_{n|n} \oplus \mathfrak{gl}_{n|n})$ and hence commutes with both Q and $F^{(\ell)}$. On V it is represented by the operator

$$\rho(\Lambda) = \sum_{j=1}^{2n} \sum_{a=1}^{N} \big(\varepsilon(e_j \otimes e_a)\iota(f_j \otimes f_a) + \mu(e_j \otimes e_a)\delta(f_j \otimes f_a) \big) ,$$

which is called the particle number in physics.

Now if τ is any positive real parameter, inserting the operator $\mathrm{e}^{-\tau\rho(\Lambda)}$ cuts off the infinite contribution from high particle numbers and makes the traces of the heuristic argument converge. Thus, on the one hand we now rigorously have

$$\mathrm{STr}_V\,\mathrm{e}^{-\tau\rho(\Lambda)}\rho(I_\ell) = \mathrm{STr}_V\,[\rho(Q), \mathrm{e}^{-\tau\rho(\Lambda)}\rho(F^{(\ell)})] = 0\ ,$$

and on the other hand, since $\rho(I_\ell) = \lambda(I_\ell) \times \mathrm{Id}_V$,

$$\mathrm{STr}_V\,\mathrm{e}^{-\tau\rho(\Lambda)}\rho(I_\ell) = \lambda(I_\ell)\,\mathrm{STr}_V\,\mathrm{e}^{-\tau\rho(\Lambda)}\ .$$

To compute the last trace, notice that $\mathrm{e}^{-\tau\Lambda}$ is the diagonal matrix

$$\mathrm{e}^{-\tau\Lambda} = \mathrm{diag}\left(\mathrm{e}^{-\tau} \times \mathrm{Id}_n, \mathrm{e}^{\tau} \times \mathrm{Id}_n, \mathrm{e}^{-\tau} \times \mathrm{Id}_n, \mathrm{e}^{\tau} \times \mathrm{Id}_n\right)$$

and use the formula (7) to obtain

$$\mathrm{STr}_V\,\mathrm{e}^{-\tau\rho(\Lambda)} = \int_{\mathrm{U}(N)} \frac{\mathrm{Det}^n(\mathrm{Id} - \mathrm{e}^{-\tau}U)\,\mathrm{Det}^n(\mathrm{Id} - \mathrm{e}^{\tau}U)}{\mathrm{Det}^n(\mathrm{Id} - \mathrm{e}^{-\tau}U)\,\mathrm{Det}^n(\mathrm{Id} - \mathrm{e}^{\tau}U)}\,dU = 1\ .$$

Therefore

$$0 = \mathrm{STr}_V\,\mathrm{e}^{-\tau\rho(\Lambda)}\rho(I_\ell) = \lambda(I_\ell)\ ,$$

and the proposition is proved. □

By combining Prop. 4.2 with the relationship between Casimir invariants and invariant differential operators, we infer:

Corollary 4.3. — *The irreducible* $\mathrm{GL}(2n|2n)$*-character* $\chi(M) = \mathrm{STr}_V\,\rho(M)$ *lies in the kernel of the full ring of* $\mathfrak{gl}_{2n|2n}$*-invariant differential operators:*

$$D(I_\ell)\chi = 0 \quad (\ell \in \mathbb{N})\ .$$

4.3. Radial differential equations for χ. — Being the character (or supertrace) of a representation, the analytic section $\chi \in \mathcal{A} := \Gamma(\mathcal{U}, \wedge E^*)$ satisfies the equation

$$(\hat{X}^L + \hat{X}^R)\chi = 0$$

for all $X \in \mathfrak{gl}_{2n|2n}$. Thus $\chi \in \mathcal{A}$ is radial, and is determined by its values as a Weyl-invariant and extendable torus function $t \mapsto \chi(t)$. Hence, if $\dot{D}(I_\ell)$ is the radial part of the invariant differential operator $D(I_\ell)$, the equation $D(I_\ell)\chi = 0$ reduces to $\dot{D}(I_\ell)\chi(t) = 0$.

We are now going to describe the radial parts $\dot{D}(I_\ell)$. For that purpose, recall from the proof of Prop. 4.1 that a maximal torus of the real-analytic domain $\mathcal{U} = \mathcal{U}^{(1)} \times \mathcal{U}^{(2)}$ is

$$T := \mathrm{U}(1)^{2n} \times \left((0,1)^n \times (1,\infty)^n\right)\ ,$$

which is parameterized by $t = \mathrm{diag}(t_1, t_2)$ with t_1 and t_2 as in (8) and variables subject to the conditions $\alpha_j, \beta_j \in \mathrm{i}\mathbb{R}$ and $\gamma_j < 0 < \delta_j$ ($j = 1, \ldots, n$). We now view these variables as coordinates on $\mathfrak{t}_\mathbb{C}$, the complexified tangent space of the flat torus T. As such they determine differential operators $\partial/\partial\alpha_j$, $\partial/\partial\beta_j$ etc. on functions $f : T \to \mathbb{C}$.

In order to transcribe Berezin's Theorem 3.7 to the present context in concise form, we need a good notation. Let us fix a set Δ_+ of positive roots of the Lie superalgebra $\mathfrak{gl}_{2n|2n}$ as follows. Arranging the coordinates as an ordered sequence

$$\delta_1, \ldots, \delta_n, \beta_1, \ldots \beta_n, \alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_n \ ,$$

Δ_+ will be taken to be the set of differences $x - y$ where x and y are any two entries from this sequence subject to the requirement that x occurs earlier than y. To illustrate: $\delta_1 - \delta_2$, $\delta_{n-1} - \beta_1$, and $\alpha_2 - \gamma_n$ are some examples of positive roots.

A root r is called *even* or *odd* depending on whether the eigenvector X in the root equation $[H, X] = r(H)X$ is an even or odd element of the Lie superalgebra. For example, in the case at hand $\delta_1 - \delta_2$ is an even root, while $\delta_{n-1} - \beta_1$ and $\alpha_2 - \gamma_n$ are odd roots. The corresponding decomposition of the system of positive roots is written $\Delta_+ = \Delta_{+,0} \cup \Delta_{+,1}$.

Proposition 4.4. — *The character $\chi : T \to \mathbb{C}$ given by $\chi(t) = \mathrm{STr}_V\, \rho(t)$ satisfies the set of differential equations $\tilde{D}(I_\ell)\chi(t) = 0$,*

$$\tilde{D}(I_\ell) = J^{-1/2} \sum_{j=1}^{n} \left(\frac{\partial^\ell}{\partial \alpha_j^\ell} + \frac{\partial^\ell}{\partial \beta_j^\ell} - (-1)^\ell \frac{\partial^\ell}{\partial \gamma_j^\ell} - (-1)^\ell \frac{\partial^\ell}{\partial \delta_j^\ell} \right) \circ J^{1/2} \ ,$$

for $\ell \in \mathbb{N}$, where the function $J^{1/2} : T \to \mathbb{C}$ is an analytic square root of

$$J(t) = \frac{\prod_{r \in \Delta_{+,0}} \sinh^2\left(\frac{1}{2} r(\ln t)\right)}{\prod_{r \in \Delta_{+,1}} \sinh^2\left(\frac{1}{2} r(\ln t)\right)} \ .$$

Proof. — On restricting the section $\chi \in \mathcal{A}$ to the torus function $\chi : T \to \mathbb{C}$, it is immediate from Cor. 4.3 that $\dot{D}(I_\ell)\chi(t) = 0$ for all $\ell \in \mathbb{N}$. By Theorem 3.7 the differential operator $\tilde{D}(I_\ell) = J^{-1/2}\tilde{D}_\ell \circ J^{1/2}$ agrees with the radial part $\dot{D}(I_\ell)$ up to lower-order terms, $J^{-1/2} P_{\ell-1} \circ J^{1/2}$. Since the lower-order operators $P_{\ell-1}$ are themselves expressed as polynomials in the $\tilde{D}_\ell$ and all constant terms $P_{\ell-1}(0,0)$ vanish (Cor. 3.9), the system of equations $\dot{D}(I_\ell)\chi = 0$ is equivalent to the system of equations $\tilde{D}(I_\ell)\chi = 0$ ($\ell \in \mathbb{N}$).

The expression for the function J follows from Theorem 3.7 on setting $p = q = 2n$ and analytically continuing from the compact torus $\mathrm{U}(1)^{4n}$ to the domain T. □

4.4. W-invariance and extendability. — Let us now spell out explicitly the properties of $\chi : T \to \mathbb{C}$ that are implied by W-invariance and extendability.

Recall from Prop. 4.1 that the character $\chi(M)$ exists as an analytic section of $\mathcal{A} = \Gamma(\mathcal{U}, \wedge E^*)$ on the domain $\mathcal{U} = \mathcal{U}^{(1)} \times \mathcal{U}^{(2)}$. Since this domain is invariant under conjugation by $g \in \mathrm{U}(2n) \times \mathrm{U}(n,n)$, so is the character:

$$\chi(M) = \chi(gMg^{-1}) \quad (\text{for all } g \in \mathrm{U}(2n) \times \mathrm{U}(n,n)) \ .$$

Proposition 4.5. — *The function* $\chi(t) = \mathrm{STr}_V\, \rho(t)$ *is invariant under the action of the Weyl group* $W = \mathrm{S}_{2n} \times (\mathrm{S}_n \times \mathrm{S}_n)$ *on the tangent space of* T*:*

$$\chi(t) = \chi(\mathrm{e}^{w \cdot \ln t}) \quad (w \in W) ,$$

where the symmetric group S_{2n} *permutes the entries in* $\ln t_1 = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ *of* $t = \mathrm{diag}(t_1, t_2)$*, while the first and second factor in the product* $\mathrm{S}_n \times \mathrm{S}_n$ *permute the first n resp. last n entries of* $\ln t_2 = (\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_n)$.

Proof. — Let $\mathfrak{t}$ be the tangent space of the flat torus T, and consider the orbit of a point $\ln t \in \mathfrak{t}$ under the adjoint action of $G := \mathrm{U}(2n) \times \mathrm{U}(n,n)$. Every distinct intersection of this orbit with $\mathfrak{t}$ gives rise to one element of the Weyl group W. Since conjugation by $g \in G$ leaves the eigenvalues of $\ln t$ unchanged, every point of intersection must correspond to a permutation of the eigenvalues. Conjugation by $(U_1, \mathrm{Id}) \in G$ with suitably chosen $U_1 \in \mathrm{U}(2n)$ allows to arbitrarily permute the $2n$ entries of $\ln t_1$. However, conjugation by (Id, U_2) with $U_2 \in \mathrm{U}(n,n)$ never mixes the sets $(\gamma_1, \ldots, \gamma_n)$ and $(\delta_1, \ldots, \delta_n)$, but only permutes the elements of these two sets separately. □

The conditions in order for a W-invariant torus function $t \mapsto f(t)$ to extend to a section $F(M)$ of $\mathcal{A}$ were discussed in Sect. 3.6. To formulate these conditions in the case at hand, we need the odd roots of the Lie superalgebra $\mathfrak{gl}_{2n|2n}$. These are all the functions $\mathfrak{t} \to \mathbb{C}$ given by

$$\alpha_i - \gamma_j , \quad \alpha_i - \delta_j , \quad \beta_i - \gamma_j , \quad \beta_i - \delta_j ,$$

for $i, j = 1, \ldots, n$. Via the nondegenerate invariant quadratic form $\langle X, Y \rangle = \mathrm{STr}\, XY$ of $\mathfrak{gl}_{2n|2n}$ restricted to $\mathfrak{t}$, each of these root functions $r : \mathfrak{t} \to \mathbb{C}$ corresponds to a vector field H_r on T by $dr = \langle H_r, \cdot \rangle$. (In Sect. 3.6 these objects were called β resp. H_β.) The latter in turn determines a first-order differential operator $\hat{H}_r$ by the directional derivative. In this way, e.g., the function $\alpha_i - \gamma_j$ translates to the differential operator $\partial/\partial\alpha_i + \partial/\partial\gamma_j$. Berezin's Theorem 3.6 now transcribes to the following.

Proposition 4.6. — *A* W*-invariant function* $f : T \to \mathbb{C}$ *extends to a section* $F \in \mathcal{A} = \Gamma(\mathcal{U}, \wedge E^*)$ *if and only if for each of the choices* $(i, j = 1, \ldots, n)$

$$\begin{array}{cc} r & \hat{H}_r \\ \alpha_i - \gamma_j , & \partial/\partial\alpha_i + \partial/\partial\gamma_j , \\ \alpha_i - \delta_j , & \partial/\partial\alpha_i + \partial/\partial\delta_j , \\ \beta_i - \gamma_j , & \partial/\partial\beta_i + \partial/\partial\gamma_j , \\ \beta_i - \delta_j , & \partial/\partial\beta_i + \partial/\partial\delta_j , \end{array}$$

$\lim_{H \to H'} r(H)^{-1} \hat{H}_r f(\mathrm{e}^H)$ *exists for all* H' *in the zero locus of* r.

Since the character $\chi(t) = \mathrm{STr}_V\, \rho(t)$ is an extendable W-invariant function, we have

Corollary 4.7. — *The function* $t \mapsto \chi(t)$ *obeys the conditions stated in Prop. 4.6.*

4.5. Proof of the main theorem. — We have now accumulated enough information about the character $t \mapsto \chi(t)$ to prove Theorem 1.1. Let us summarize everything that we know.

A. χ is an analytic function on the torus
$$T = \mathrm{U}(1)^{2n} \times (0,1)^n \times (1,\infty)^n ,$$
which we parameterize by $\alpha_j, \beta_j \in \mathrm{i}\mathbb{R}$ and $\gamma_j < 0 < \delta_j$ $(j = 1, \ldots, n)$ as in (8).

B. In the limiting situation where the variables δ_j and γ_j $(j = 1, \ldots, n)$ are sent to $\pm\infty$, the function χ has the asymptotic behavior given in Prop. 2.5.

C. χ lies in the kernel of the ring of invariant differential operators; by Prop. 4.4 this is equivalent to the system of equations $\tilde{D}(I_\ell)\chi(t) = 0$ $(\ell \in \mathbb{N})$.

D. The W-invariant and extendable function χ has the properties specified in Props. 4.5 and 4.6.

The proof of Theorem 1.1 is done in two steps: we first show that the conditions (A-D) admit at most one solution χ; afterwards we will write down the solution and verify that it has the required properties.

4.5.1. Uniqueness of the solution. — To prove uniqueness, we shall demonstrate that the solution can be constructed in the form of a convergent power series. The starting point of this construction is Prop. 2.5: sending the parameters δ_j and γ_j to $\pm\infty$, we know the asymptotics of χ to be $\chi/\chi_\infty \to 1$ with
$$\chi_\infty(t) = \sum_{[w] \in W/W_\lambda} \chi_\infty^\lambda(\mathrm{e}^{w^{-1}\cdot \ln t}) , \quad \chi_\infty^\lambda(t) = \frac{\mathrm{e}^{N\lambda(\ln t)}}{\prod_{i,j}(1 - \mathrm{e}^{(\alpha_i - \beta_j)(\ln t)})} .$$
Here $W = \mathrm{S}_{2n} \times (\mathrm{S}_n \times \mathrm{S}_n)$ is the Weyl group of our problem (as before in Prop. 4.5), and $W_\lambda \subset W$ denotes the subgroup that stabilizes
$$\lambda := \textstyle\sum_j (\beta_j - \delta_j) .$$

χ_∞ is the leading term in a power series for the analytic function χ. Actually, each summand in the formula for χ_∞ is the leading term of such a series, and to construct the function χ it is enough to construct the power series built on one of these terms, say χ_∞^λ. The complete answer for χ is then obtained by summing over W-translates.

Our procedure will therefore be to look for solutions of the system of differential equations $\tilde{D}(I_\ell)\chi^\lambda(t) = 0$ $(\ell \in \mathbb{N})$ with the asymptotic data given by $\chi^\lambda/\chi_\infty^\lambda \to 1$. If χ^λ is such a solution, then since the operators $\tilde{D}(I_\ell)$ are W-invariant, the sum
$$\chi(t) = \sum_{[w] \in W/W_\lambda} \chi^\lambda(\mathrm{e}^{w^{-1}\cdot \ln t})$$
will be a W-invariant solution of the system $\tilde{D}(I_\ell)\chi(t) = 0$ with the required limit at infinity. Conversely, every analytic solution χ of this system with the prescribed asymptotics can be presented as such a sum.

To tackle the construction of a power series for χ^λ, recall $\lambda = \sum_j(\beta_j - \delta_j)$. The significance of this linear function is that $N\lambda$ plays the role of (non-dominant) highest weight for the representation (V,ρ). Given λ, define

$$\Delta_\lambda := \{r \in \Delta_+ \mid \langle r,\lambda\rangle = 0\} .$$

Thus Δ_λ is the subset of positive roots which have vanishing scalar product with $N\lambda$. The function J of Prop. 4.4 now decomposes as

$$J = J_\lambda \times (J/J_\lambda) =: J_\lambda \times j_\lambda ,$$

where J_λ is given by the same formula as J but with the products in the numerator and denominator restricted to run over the subsystem of roots $\Delta_\lambda \subset \Delta_+$:

$$J_\lambda(t) = \frac{\prod_{r\in\Delta_{\lambda,0}} \sinh^2\left(\frac{1}{2}r(\ln t)\right)}{\prod_{r\in\Delta_{\lambda,1}} \sinh^2\left(\frac{1}{2}r(\ln t)\right)} .$$

Definition 4.8. — *Take Λ_+ to be the integer lattice of linear functions*

$$\mu = \sum_{j=1}^n (a_j\alpha_j - b_j\beta_j + c_j\gamma_j - d_j\delta_j)$$

with a_j, b_j, c_j, and d_j in $\mathbb{N}\cup\{0\}$, and $\sum_{j=1}^n(c_j+d_j) > 0$.

Now recall the formula $\tilde{D}(I_\ell) = J^{-1/2}\tilde{D}_\ell \circ J^{1/2}$ and consider the system of equations

$$0 = \tilde{D}(I_\ell)\chi^\lambda = \tilde{D}_\ell\left(J^{1/2}\chi^\lambda\right) \quad (\ell \in \mathbb{N}) .$$

If $\pi_s := \prod_{i,j}(1-\mathrm{e}^{\alpha_i-\beta_j}) \circ \ln$ is the denominator of $\chi^\lambda_\infty = \mathrm{e}^{N\lambda}/\pi_s$, write

$$J^{1/2}\chi^\lambda = J_\lambda^{1/2} \times \left(j_\lambda^{1/2}/\pi_s\right) \times \left(\pi_s\chi^\lambda\right) ,$$

and notice that the middle factor,

$$j_\lambda^{1/2}/\pi_s = \prod_{i,j} \frac{1-\mathrm{e}^{\gamma_k-\delta_l}}{(1-\mathrm{e}^{\alpha_k-\delta_l})(1-\mathrm{e}^{\gamma_i-\beta_j})} \circ \ln ,$$

possesses a power series expansion:

$$j_\lambda^{1/2}(t)/\pi_s(t) = 1 + \sum_{\mu\in\Lambda_+} B(\mu)\,\mathrm{e}^{-\mu(\ln t)} ,$$

which converges everywhere on T. It is therefore possible to look for the analytic function χ^λ in the form of a power series

$$\chi^\lambda(t) = \chi^\lambda_\infty(t)\left(1 + \sum_{\mu\in\Lambda_+} C(\mu)\,\mathrm{e}^{-\mu(\ln t)}\right) . \tag{9}$$

Lemma 4.9. — *If the system of differential equations $\tilde{D}(I_\ell)\chi^\lambda(t) = 0$ ($\ell \in \mathbb{N}$) has a solution χ^λ with limit $(\chi^\lambda/\chi^\lambda_\infty) \to 1$ at infinity, then this solution is unique.*

Proof. — Set $C(0) = B(0) = 1$. Inserting the ansatz (9) into the system of differential equations we then have

$$\forall \ell \in \mathbb{N}: \quad \sum_{\mu \in \Lambda_+ \cup \{0\}} \left(\sum_{\nu \in \Lambda_+ \cup \{0\}} B(\nu) C(\mu - \nu) \right) \tilde{D}_\ell \left(J_\lambda^{1/2} \mathrm{e}^{N\lambda - \mu} \right) = 0 \, .$$

The functions $t \mapsto J_\lambda(t)^{1/2} \mathrm{e}^{(N\lambda - \mu)(\ln t)}$ for $\mu \in \Lambda_+$ are all linearly independent. This statement remains true after application of the differential operators $\tilde{D}_\ell$. Indeed, moving the variables α_j, β_j off the imaginary axis we may expand the sinh-functions in $J_\lambda{}^{1/2}$ into convergent series of exponential functions. The resulting exponentials are eigenfunctions of the differential operators (with constant coefficients) $\tilde{D}_\ell$; and for $\nu \in \Lambda_+$ none of eigenvalues vanishes for all $\ell \in \mathbb{N}$ simultaneously.

It now follows that the coefficients $C(\mu)$ of any solution χ^λ must satisfy

$$\forall \mu \in \Lambda_+: \quad C(\mu) + \sum_{\nu \in \Lambda_+} B(\nu) C(\mu - \nu) = 0 \, ,$$

which constitutes a recursive system of equations for the unknowns $C(\mu)$. Its solution is unique once the limit $\chi^\lambda / \chi^\lambda_\infty \to 1 = C(0)$ has been fixed. □

4.5.2. Existence of solution. — To restate the result claimed in Theorem 1.1, let S_{W/W_λ} be the symmetrization operator

$$S_{W/W_\lambda} f(t) := \sum_{[w] \in W/W_\lambda} f(\mathrm{e}^{w^{-1} \cdot \ln t}) \, ,$$

which is defined for the case of functions $f \circ \exp : \mathfrak{t} \to \mathbb{C}$ that are W_λ-invariant.

Lemma 4.10. — *A solution to the problem posed by the conditions (A–D) above is*

$$\chi(t) = S_{W/W_\lambda} \left(\mathrm{e}^{N\lambda(\ln t)} j_\lambda^{-1/2}(t) \right) \, ,$$

where $j_\lambda^{-1/2}$ is taken to be the analytic square root

$$j_\lambda^{-1/2} = \prod_{k,l=1}^{n} \frac{\sinh\left(\frac{1}{2}(\alpha_k - \delta_l)\right) \sinh\left(\frac{1}{2}(\gamma_k - \beta_l)\right)}{\sinh\left(\frac{1}{2}(\alpha_k - \beta_l)\right) \sinh\left(\frac{1}{2}(\gamma_k - \delta_l)\right)} \circ \ln \, .$$

Proof. — Let us verify the properties A–D in sequence.

The function $\mathrm{e}^{N\lambda} j_\lambda^{-1/2}$ approaches χ^λ_∞ at infinity for δ_l and γ_l ($l = 1, \ldots, n$), and is analytic on a dense open subset in T. The apparent singularities from $\alpha_k - \beta_l = 0$ (mod 2πi) are "healed" for χ by the symmetrization S_{W/W_λ}. Thus properties A and B are fulfilled.

Turning to C, since the radial differential operators $\tilde{D}(I_\ell)$ commute with symmetrization operator S_{W/W_λ}, we have

$$0 = \tilde{D}(I_\ell)\chi = S_{W/W_\lambda} \left(J^{-1/2} \tilde{D}_\ell \circ J_\lambda^{1/2} \mathrm{e}^{N\lambda} \right) \, .$$

To establish property C, it therefore suffices to show that

$$\tilde{D}_\ell \left(J_\lambda^{1/2} \mathrm{e}^{N\lambda} \right) = 0 \,.$$

For this purpose, observe that the even roots in Δ_λ are

$$\alpha_i - \alpha_j \,, \quad \beta_i - \beta_j \,, \quad \gamma_i - \gamma_j \,, \quad \delta_i - \delta_j \quad (1 \le i < j \le n) \,,$$

while the odd roots in that subsystem are $\alpha_i - \gamma_j$ and $\beta_i - \delta_j$ ($i, j = 1, \ldots, n$). Thus

$$\begin{aligned} J_\lambda &= \frac{\prod_{i<j} \sinh\left(\frac{1}{2}(\alpha_i - \alpha_j)\right) \sinh\left(\frac{1}{2}(\gamma_i - \gamma_j)\right)}{\prod_{i,j} \sinh\left(\frac{1}{2}(\alpha_i - \gamma_j)\right)} \\ &\times \frac{\prod_{i<j} \sinh\left(\frac{1}{2}(\beta_i - \beta_j)\right) \sinh\left(\frac{1}{2}(\delta_i - \delta_j)\right)}{\prod_{i,j} \sinh\left(\frac{1}{2}(\beta_i - \delta_j)\right)} \,. \end{aligned}$$

By Lemma 3.8 this can be expressed more simply as a product of two determinants:

$$\begin{aligned} J_\lambda &= \mathrm{Det}\left(\sinh^{-1}\left(\tfrac{1}{2}(\alpha_i - \gamma_j)\right)\right)_{i,j=1,\ldots,n} \\ &\times \mathrm{Det}\left(\sinh^{-1}\left(\tfrac{1}{2}(\beta_i - \delta_j)\right)\right)_{i,j=1,\ldots,n} \,. \end{aligned}$$

By expanding the reciprocals of the sinh-functions into convergent power series,

$$\frac{1/2}{\sinh\left(\frac{1}{2}(\alpha_i - \gamma_j)\right)} = \frac{\mathrm{e}^{\frac{1}{2}(\gamma_j - \alpha_i)}}{1 - \mathrm{e}^{\gamma_j - \alpha_i}} = \mathrm{e}^{\frac{1}{2}(\gamma_j - \alpha_i)} + \mathrm{e}^{\frac{3}{2}(\gamma_j - \alpha_i)} + \ldots,$$

it is now easy to verify that $\tilde{D}_\ell\left(J_\lambda^{1/2} \mathrm{e}^{N\lambda}\right) = 0$ indeed holds for all $\ell \in \mathbb{N}$.

Finally, W-invariance (property D) of χ is not an issue, as this is enforced by the symmetrizer S_{W/W_λ}. It remains to verify the conditions for extendability which are listed in Prop. 4.6. To show that χ satisfies these regularity conditions, we first show that $j_\lambda^{-1/2}$ satisfies them. Fix some pair $k, l \in \{1, \ldots, n\}$ and apply to $j_\lambda^{-1/2}$ the differential operator associated with, e.g., the root $\beta_k - \gamma_l$:

$$\begin{aligned} \frac{2}{\beta_k - \gamma_l} \left(\frac{\partial}{\partial \beta_k} + \frac{\partial}{\partial \gamma_l} \right) j_\lambda^{-1/2} &= \frac{j_\lambda^{-1/2}}{\beta_k - \gamma_l} \sum_{i=1}^{n} \left\{ \coth\left(\tfrac{1}{2}(\beta_k - \gamma_i)\right) - \coth\left(\tfrac{1}{2}(\beta_k - \alpha_i)\right) \right. \\ &\qquad \left. + \coth\left(\tfrac{1}{2}(\gamma_l - \beta_i)\right) - \coth\left(\tfrac{1}{2}(\gamma_l - \delta_i)\right) \right\} . \end{aligned}$$

After summation over i, the expression in curly brackets is regular at the zero locus of $\beta_k - \gamma_l$, since there are only two terms which are singular and these cancel each other:

$$\coth\left(\tfrac{1}{2}(\beta_k - \gamma_l)\right) + \coth\left(\tfrac{1}{2}(\gamma_l - \beta_k)\right) = 0 \,.$$

Regularity of the expression above is then ensured by the fact that $\beta_k - \gamma_l$ divides $j_\lambda^{-1/2}$, i.e., $j_\lambda^{-1/2}$ has a zero, $j_\lambda^{-1/2} \sim \sinh\left(\frac{1}{2}(\beta_k - \gamma_l)\right)$, at the zero locus of $\beta_k - \gamma_l$. The same argument holds for the roots $\alpha_k - \delta_l$. For the roots $\alpha_k - \gamma_l$ and $\beta_k - \delta_l$ a similar argument goes through, except that these roots do not divide $j_\lambda^{-1/2}$ but divide the corresponding expression in curly brackets.

Regularity still holds after multiplying $j_\lambda^{-1/2}$ by $e^{N\lambda} = e^{N\sum_i(\beta_i-\delta_i)}$. For the roots $\alpha_k - \gamma_l$ this is obvious, for the roots $\alpha_k - \delta_l$ and $\beta_k - \gamma_l$ it follows because they divide $j_\lambda^{-1/2}$, and for $\beta_k - \delta_l$ it follows because the differential operator $\partial/\partial\beta_k + \partial/\partial\delta_l$ annihilates the factor $e^{N\sum_i(\beta_i-\delta_i)}$.

Since the Weyl group W just permutes the odd roots, regularity persists after application of the symmetrizer S_{W/W_λ}. Thus we have verified property D. $\square$

Our solution to the problem posed by the conditions (A-D) is easily seen to coincide with the one stated in Theorem 1.1. Since the solution is unique by Lemma 4.9, the proof of that theorem is now complete.

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