Cosine Products, Fourier Transforms, and Random Sums

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1. INTRODUCTION. The function $\sin x/x$ is endlessly fascinating. By setting $x = \pi/2$ in the infinite product expansion

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \cos\frac{x}{2^k} \tag{1}$$

one gets the first actual formula for π that mankind ever discovered, dating from 1593 and due to François Viète (1540–1603), whose Latinized name is Vieta. (Was any notice taken of the formula's 400th anniversary, perhaps by the issue of a postage stamp?) From the samples of a function $f(x)$ at equally spaced points x_n, $n \in \mathbf{Z}$, one can reconstruct the complete function with the aid of $\sin x/x$, provided f is "band-limited" and the spacing of the samples is small enough. This is the content of the Sampling Theorem, which lends its name to $\sin x/x$ as the **sampling function**. Its importance in signal processing, where it is also known as sinc x, is the result of its Fourier transform being the characteristic function of the interval $[-1, 1]$ (modulo a scalar factor).

In Section 2 we prove the infinite product expansion for $\sin x/x$ and derive Viète's formula. In Section 3 we transform the product expansion with the Fourier transform and use convolution and delta distributions to prove it in a way that reveals a host of similar identities. Section 4 puts these identities into a probabilistic setting, and in Section 5 we alter the probability experiments in order to make connections between infinite cosine products, Cantor sets, and sums of series with random signs, particularly the harmonic series. This leaves us with some interesting unsolved problems and conjectures for further work.

2. AN ELEMENTARY PROOF. Repeated use of the double angle formula for the sine shows that

$$\begin{aligned} \sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} \\ &= 4\sin\frac{x}{4}\cos\frac{x}{4}\cos\frac{x}{2} \\ &\;\;\vdots \\ &= 2^n \sin\frac{x}{2^n}\left(\prod_{k=1}^{n}\cos\frac{x}{2^k}\right). \end{aligned}$$

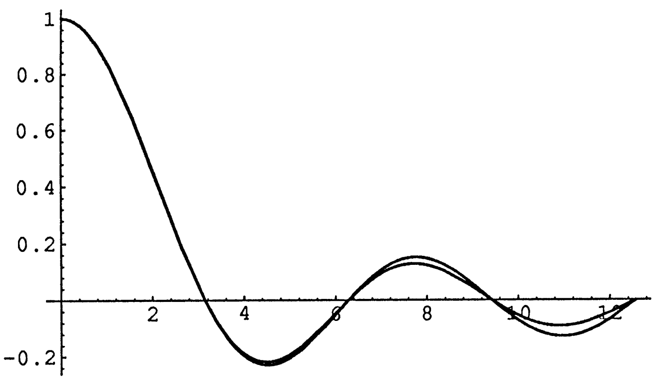

Figure 1. Graphs of $\sin x/x$ and $\cos x/2 \cos x/4 \cos x/8$. Where both graphs are visible, $\sin x/x$ is nearer the x axis.

But

$$\lim_{n\to\infty} 2^n \sin\frac{x}{2^n} = x,$$

thereby proving the identity. See Figure 1 for an indication of how quickly the product converges.

Let $x = \pi/2$, make use of the half-angle identity, and there you have Viète's formula for π,

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\,\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots. \tag{2}$$

At this point the cosine identity could remain an isolated curiosity of historical interest, relegated to the ends of exercise sets in textbooks. In fact, it is just the first of an infinite family of cosine product identities for $\sin x/x$.

3. THE FOURIER TRANSFORM AND MORE IDENTITIES. For a complex valued function $f(x)$ defined on the real line, the Fourier transform puts together f as a continuous linear combination of the "pure" oscillations $e^{i\omega x}$ in which the coefficient in front of $e^{i\omega x}$ is denoted by $\hat{f}(\omega)$. Thus,

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega. \tag{3}$$

The function $\hat{f}$ is the **Fourier transform** of f and the integral above is a description of how to get back f from $\hat{f}$ and is actually the formula for the inverse transform. How do we get $\hat{f}$ from f? That is given by this integral:

$$\hat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx. \tag{4}$$

Of course, the proofs of these relationships involve hypotheses on the functions so that the integrals make sense, but they can be extended beyond the realm of ordinary functions to generalized functions or distributions. We need more than ordinary functions in order to make sense of the Fourier transform of a sine or cosine.

Notation: we also write the Fourier transform of f as $\mathscr{F}(f)$ and the inverse transform of ϕ as $\mathscr{F}^{-1}(\phi)$.

Consider $\cos bx$, which by Euler's Identity may be written as

$$\cos bx = \tfrac{1}{2}\left(e^{ibx} + e^{-ibx}\right).$$

This shows the function written as a linear combination of just two of the functions $e^{i\omega x}$ for $\omega = b$ and $\omega = -b$. The coefficients appear to be $1/2$, but if we use them in the integral form with all other coefficients zero, then we cannot represent the cosine function. Instead, we must regard the coefficients as point masses at b and $-b$. Therefore, the Fourier transform of $\cos bx$ is $(1/2)(\delta_b + \delta_{-b})$, where δ_b denotes the Dirac delta distribution or point mass at the point b. All of this can be made rigorous, but at the expense of some long development in graduate level analysis. The approach here is at about the level of a second year course in engineering mathematics.

In addition, the Fourier transform behaves nicely on a product of functions and turns it into the convolution of the transforms:

$$\widehat{fg} = \hat{f} * \hat{g}. \tag{5}$$

For two functions $\phi(\omega)$ and $\psi(\omega)$, the convolution $\phi * \psi$ is defined by

$$(\phi * \psi)(\omega) = \int_{-\infty}^{\infty} \phi(\alpha)\psi(\omega - \alpha)\, d\alpha. \tag{6}$$

Again, we must extend convolution beyond the realm of functions. In particular we need convolutions of delta distributions and for them we can easily show that δ_0 behaves as the identity for convolution

$$\delta_0 * \phi = \phi \tag{7}$$

and that

$$\delta_a * \delta_b = \delta_{a+b}. \tag{8}$$

Now back to the cosine identity. Let $f(x) = \prod_{k=1}^{\infty} \cos(x/2^k)$ and let f_n be the nth partial product. The Fourier transform of f_n is

$$\hat{f}_n = *\prod_{k=1}^{n} \frac{1}{2}\left(\delta_{1/2^n} + \delta_{-1/2^n}\right).$$

The asterisk in front of the product sign indicates a repeated convolution of the factors. Expanding for $n = 3$ we see that

$$\hat{f}_3 = \tfrac{1}{8}\left(\delta_{-7/8} + \delta_{-5/8} + \cdots \delta_{7/8}\right).$$

Likewise

$$\hat{f}_n = \frac{1}{2^n} \sum_{b \in B_n} \delta_b,$$

where B_n is the set of 2^n equally spaced numbers from $-1 + 1/2^n$ to $1 - 1/2^n$ with spacing $2/2^n = 1/2^{(n-1)}$.

The sequence of measures $\hat{f}_n$ converges to the uniform density on $[-1, 1]$ of total mass 1, which we can write as $(1/2)\chi_{[-1,1]}\, d\omega$. The inverse transform is easy to compute:

$$\int_{-1}^{1} \frac{1}{2} e^{i\omega x}\, d\omega = \frac{\sin x}{x}.$$

The spectrum of $(\sin x)/x$ is uniform in the interval $-1 \le \omega \le 1$. This means that $\sin x/x$ is a continuous linear combination of the "pure" harmonics $e^{i\omega x}$ with the same weight of $1/2$ for each $\omega \in [-1, 1]$.

With this proof we have a way to generate a family of similar identities. Let us put point masses at 3^n equally spaced points from $-1 + 1/3^n$ to $1 - 1/3^n$ with spacing $2/3^n$. Such a measure is the convolution $*\prod_{k=1}^{n} \frac{1}{3}(\delta_{-2/3^k} - \delta_0 + \delta_{2/3^k})$. Applying the inverse transform

should be +

$$\mathscr{F}^{-1}\left(\frac{1}{3}(\delta_{-2/3^k} + \delta_0 + \delta_{2/3^k})\right) = \frac{1}{3}\left(2\cos\frac{2x}{3^k} + 1\right)$$

and taking limits gives us the infinite product identity

$$\prod_{n=1}^{\infty} \frac{1}{3}\left(1 + 2\cos\frac{2x}{3^n}\right) = \frac{\sin x}{x}. \tag{9}$$

Let us use the positive integer p as the base (we have just seen $p = 2$ and $p = 3$). The first measure $\hat{f}_1$ is the sum of point masses at p points equally spaced from $-1 + 1/p$ to $1 - 1/p$ with spacing $2/p$.

$$\hat{f}_1 = \frac{1}{p}\left(\delta_{\frac{1-p}{p}} + \delta_{\frac{3-p}{p}} + \delta_{\frac{5-p}{p}} + \cdots + \delta_{\frac{p-1}{p}}\right) \tag{10}$$

$$= \frac{1}{p}\sum_{l=0}^{p-1} \delta_{\frac{2l+1-p}{p}} \tag{11}$$

We let

$$\hat{f}_n = *\prod_{k=1}^{n}\left(\frac{1}{p}\sum_{l=0}^{p-1} \delta_{\frac{2l+1-p}{p^k}}\right)$$

and one can see that $\hat{f}_n$ consists of p^n point masses equally spaced from $-1 + 1/p^n$ to $1 - 1/p^n$ with spacing $2/p^n$. Taking the inverse transform we see that

$$\mathscr{F}^{-1}(\hat{f}_n)(x) = \prod_{k=1}^{n} \frac{1}{p}\sum_{l=0}^{p-1} \exp((2l + 1 - p)ix/p^k).$$

Rewriting the exponential as cosines and taking limits gives the general identities.

There is a slight difference in the form depending on the parity of p. For p even

$$\prod_{k=1}^{\infty} \frac{1}{p}\left(\sum_{\substack{1 \le m \le p-1 \\ m \text{ odd}}} 2\cos\frac{mx}{p^k}\right) = \frac{\sin x}{x}. \tag{12}$$

For p odd

$$\prod_{k=1}^{\infty} \frac{1}{p}\left(1 + \sum_{\substack{1 \le m \le p-1 \\ m \text{ even}}} 2\cos\frac{mx}{p^k}\right) = \frac{\sin x}{x}. \tag{13}$$

For $p = 6$ the identity takes the form

$$\prod_{k=1}^{\infty} \frac{1}{6}\left(2\cos\frac{x}{6^k} + 2\cos\frac{3x}{6^k} + 2\cos\frac{5x}{6^k}\right) = \frac{\sin x}{x}. \tag{14}$$

For $p = 7$ the identity takes the form

$$\prod_{k=1}^{\infty} \frac{1}{7}\left(1 + 2\cos\frac{2x}{7^k} + 2\cos\frac{4x}{7^k} + 2\cos\frac{6x}{7^k}\right) = \frac{\sin x}{x}. \tag{15}$$

For larger p fewer terms in the product are needed for the same degree of accuracy in the approximation to $\sin x/x$. In fact, by letting p go to infinity the first factor alone approaches $\sin x/x$ and provides a novel derivation of a well-known result. I leave it to the reader to work it out.

4. PROBABILISTIC INTERPRETATION. Mark Kac, in his delightful and now classic Carus monograph [**2**], proves the first cosine identity (1) in a way that is equivalent to the one we have outlined, although he does not explicitly use the Fourier transform, delta functions, and convolution. He then turns the identity into a question of probability, which for him was the leitmotif of his mathematical work.

The original product identity (1) arises from the following experiment. Flip a fair coin repeatedly. Beginning with 0, add $1/2$ if the result is heads and subtract $1/2$ if the result is tails. On the next toss add or subtract $1/4$; on the next add or subtract $1/8$, and so on. What is the distribution of the sums over the probability space whose elements are the countable sequences of coin tosses? Clearly the sums are distributed uniformly between -1 and 1.

Let s_n denote the nth partial sum. It is a sum of independent random variables $a_1 + a_2 + \cdots + a_n$, where a_k has the probability distribution $(1/2)(\delta_{1/2^k} + \delta_{-1/2^k})$. The probability distribution of a sum of independent random variables is the convolution of the respective distributions of the random variables. Therefore, s_n has the distribution

$$*\prod_{k=1}^{n} \frac{1}{2}\left(\delta_{1/2^k} + \delta_{-1/2^k}\right) = \frac{1}{2^n}\left(\delta_{\frac{1-2^n}{2^n}} + \cdots + \delta_{\frac{2^n-1}{2^n}}\right).$$

The inverse Fourier transform of a probability measure is called its **characteristic function**. Thus, the characteristic function for the distribution of s_n is the product $\prod_{k=1}^{n} \cos x/2^k$. In the theory of probability and statistics, characteristic functions are a powerful tool. Typically computations are done with characteristic functions in order to draw conclusions about distributions of random variables as in the standard proof of the Central Limit Theorem. Here, however, we have inverted the relationship in order to compute with the probability measures and to get results about the characteristic functions.

5. RELATED PRODUCTS: EXAMPLES AND CONJECTURES

5.1. Coin tossing and Cantor sets. The Cantor set K is the set of points between 0 and 1 whose ternary expansion has no 1's in it. So z is in K if $z = \sum_{k=1}^{\infty} t_k 3^{-k}$, $t_k \in \{0, 2\}$. Define K_n to be the set of elements of K that have the form $\sum_{k=1}^{n} t_k 3^{-k}$, and define a probability measure supported on K_n

$$\mu_n = \frac{1}{2^n} \sum_{z \in K_n} \delta_z. \tag{16}$$

K_n has 2^n elements so μ_n is equally distributed on K_n. The sequence (μ_n) has a limit μ, which can be described as assigning the following limit as the measure of a set E:

$$\mu(E) = \lim_{n\to\infty} \frac{\#E \cap K_n}{2^n}. \tag{17}$$

The measure μ is also the Lebesgue-Stieltjes measure of the Cantor function. The Cantor function is continuous, non-decreasing, and has derivative zero on the complement of the Cantor set. Thus it defines a measure supported on the Cantor set, which is precisely the measure μ defined in (17).

What is of interest in this note is that μ_n is the finite convolution product

$$\mu_n = *\prod_{k=1}^{n} \frac{1}{2}(\delta_0 + \delta_{2/3^k}). \tag{18}$$

Consider the experiment of tossing a fair coin. On toss number k let

$$a_k = \begin{cases} 0 & \text{heads} \\ 2/3^k & \text{tails} \end{cases}$$

Let $s_n = \sum_{k=1}^{n} a_k$. Then s_n is equally distributed over K_n. The characteristic function for the distribution of s_n is $\prod_{k=1}^{n}(1/2)(1 + e^{2xi/3^k})$. Define

$$f(x) = \prod_{k=1}^{\infty} \frac{1}{2}(1 + e^{2xi/3^k}). \tag{19}$$

(One checks easily that the product is convergent.) Then $\hat{f} = \mu$, the Cantor measure, but is it possible to characterize f in any other way?

This leads us to look at the related infinite product $\prod_{k=1}^{\infty} \cos 2x/3^k$. Because

$$\mathscr{F}\left(\cos\frac{2x}{3^k}\right) = \frac{1}{2}(\delta_{2/3^k} + \delta_{-2/3^k})$$

the probabilistic interpretation is clear: add or subtract $2/3^k$ on the kth toss with equal probability. Let s_n be the sum of the first n values. What is the distribution of s_n and what is the distribution of $s = \lim_{n\to\infty} s_n$? The exercise of expanding and plotting the values of s_3 lead one to suspect that s is distributed "uniformly" over the Cantor set constructed from $[-1, 1]$ by successively removing middle thirds. That is easy to prove, as follows.

Define the affine map of $[0, 1]$ to $[-1, 1]$ by $z \mapsto 2(z - 1/2)$. Let $z = \sum t_k 3^{-k}$, $t_k \in \{0, 2\}$, be a point in the Cantor set. The ternary expansion of $1/2$ is $\sum 3^{-k}$, and so $2(z - 1/2) = \sum 2(t_k - 1)3^{-k}$. The coefficients $2(t_k - 1)$ are either 2 or -2 with equal probability.

This shows that the infinite product $\prod_{k=1}^{\infty} \cos 2x/3^k$ has Fourier transform equal to the Cantor measure on the Cantor set constructed from $[-1, 1]$ by removing middle thirds, but it does not give us a closed form like $\sin x/x$. It would be most surprising if there were any simpler description of $\prod_{k=1}^{\infty} \cos 2x/3^k$. In Figure 2 is a plot of the partial product with $n = 8$ and $0 \le x \le 100$. (The function is even.) Over this range the infinite product is indistinguishable from the eighth partial product. The self-similarity of the Cantor set at smaller and smaller scales appears to be reflected in the self-similarity of the graph at higher and higher frequencies.

5.2. Harmonic Series with Random Signs. We have been looking at the sums of series of the form

$$\sum_{k=1}^{\infty} t_k c_k \tag{20}$$

where t_k is randomly chosen to be 1 or -1 with equal probability. Rademacher proved that if $\sum c_k^2 < \infty$, then the sum converges with probability one on the probability space $\Omega = \{-1, 1\}^N$. (Ω can be identified with the unit interval and the

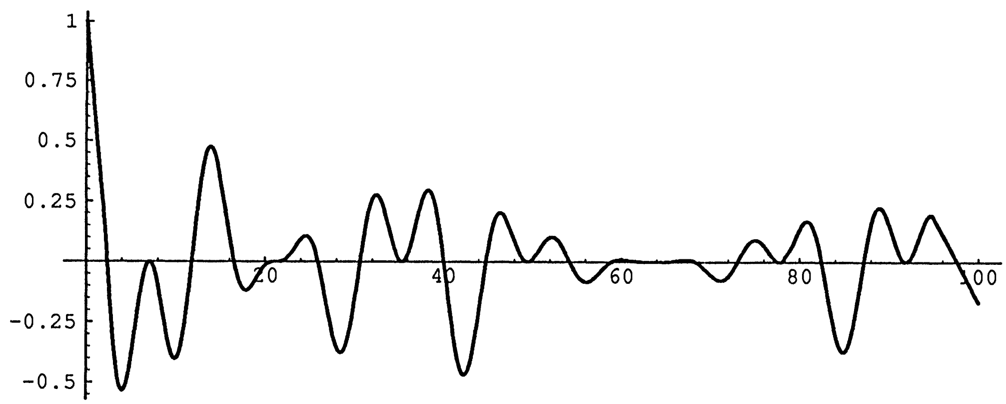

Figure 2. Graph of $\prod_{k=1}^{8} \cos \frac{2x}{3^k}$.

probability measure with Lebesgue measure by using binary representations of numbers in the interval.) In [**2**] Kac gives the proof of this theorem due to Paley and Zygmund. It is also a theorem that the series diverges with probability one if $\Sigma c_k^2 = \infty$. Let us consider the random harmonic series

$$\sum_{k=1}^{\infty} \frac{t_k}{k}, \tag{21}$$

which converges almost surely by Rademacher's result, with the goal of understanding the distribution of the sums. This means we want to understand the distribution of the random variable s defined on Ω. If we let s_n be the partial sum, also a random variable, then the probability distribution of s_n is the measure

$$\mu_n = * \prod_{k=1}^{n} \frac{1}{2}(\delta_{1/k} + \delta_{-1/k}) \tag{22}$$

and its inverse transform is

$$\mathscr{F}^{-1}(\mu)(x) = \prod_{k=1}^{n} \cos \frac{x}{k}. \tag{23}$$

The product converges uniformly on compact sets as $n \to \infty$, and so it is plausible that the sequence μ_n converges to a probability measure μ that is the distribution of the random variable s. There is, however, a fair bit of analysis to make this rigorous. Assuming that the analysis can be made rigorous, then the plot of the Fourier transform of the infinite product $\prod_{k=1}^{\infty} \cos x/k$ will show how the sums are distributed. Let us call this function $\phi(\omega)$. Then

$$\phi(\omega) = \mathscr{F}\left(\prod_{k=1}^{\infty} \cos \frac{x}{k}\right)(\omega) \tag{24}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x} \prod_{k=1}^{\infty} \cos \frac{x}{k}\, dx \tag{25}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} (\cos \omega x + i \sin \omega x) \prod_{k=1}^{\infty} \cos \frac{x}{k}\, dx \tag{26}$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \cos \omega x \prod_{k=1}^{\infty} \cos \frac{x}{k}\, dx. \tag{27}$$

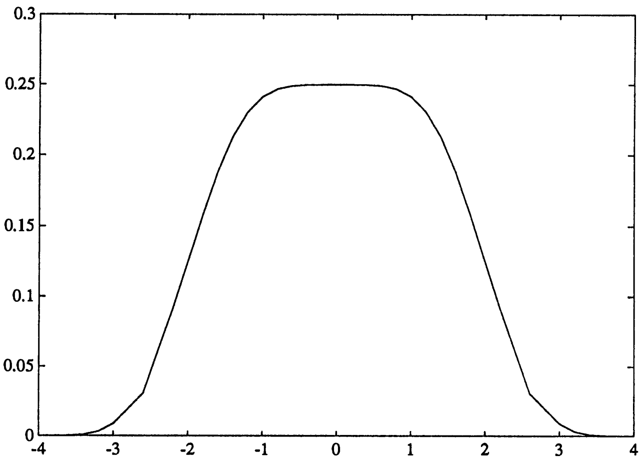

Figure 3. Graph of $\phi(\omega)$.

There is not a closed form for $\phi(\omega)$ and so we resort to numerical integration. We truncated the infinite product at $n = 1000$ and integrated from 0 to 15 using a straightforward Riemann sum with $dx = 0.02$ and the midpoints of the subintervals for the points of evaluation. Values for ω were from 0 to 3.8 in multiples of 0.2. The integration was done with True BASIC on a portable Macintosh. See Figure 3. The distribution is very flat for $-1 < \omega < 1$, much flatter than a normal distribution. A few of the computed values are given in this table. The value of $\phi(0)$ is suspiciously close to $1/4$, suggesting perhaps that $\pi/4$ is the value of the integral

$$\int_0^\infty \prod_{k=1}^{\infty} \cos\frac{x}{k}\, dx. \tag{28}$$

One might also conjecture that $\int_0^\infty \cos 2x \prod_{k=1}^\infty \cos x/k\, dx = \pi/8$.

For additional evidence we turned to simulations of the sums. Using MATLAB we ran 5000 sums of $\sum_{k=1}^{100} t_k/k$ with the values of t_k picked randomly as ± 1 with equal probability. Figure 4 shows a histogram of the sums.

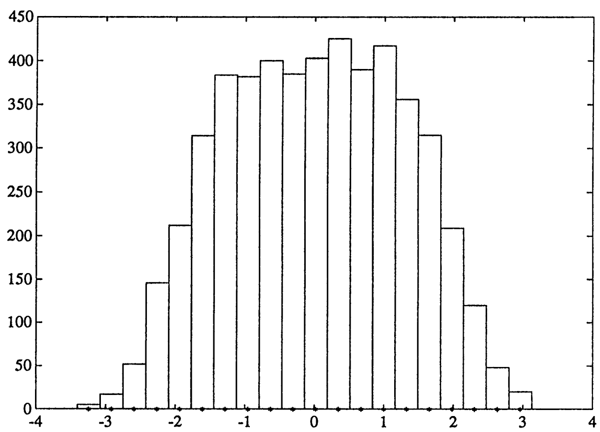

Figure 4. Histogram of 5000 random sums.

ω	$\phi(\omega)$
0.0	.249995
0.1	.249991
0.2	.249972
0.4	.249809
0.6	.249092
0.8	.246819
1.0	.241289
1.2	.230494
1.4	.212941
1.6	.188425
1.8	.158271
2.0	.125000
2.2	.091729
2.4	.061576
2.6	.030596
2.8	.019506
3.0	.008711
3.2	.003181
3.4	.000908
3.6	.000192
3.8	.000028

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Proof requires a person who can give and a person who can receive . . .

—*Augustus De Morgan (1808–1871)*

Budget of Paradoxes. London: 1872, p. 262.