BEYOND ENDOSCOPY: ITS CONNECTION TO L-FUNCTIONS AND THE FUNDAMENTAL LEMMA.

RESEARCH STATEMENT
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One tool a number theorist often uses is an L-function. L-functions arise from many arithmetic objects of interest—such as elliptic curves, Galois groups, and Dirichlet characters. The L-functions coming from Galois groups or in particular, from representations of Galois groups are especially sought after. One famous conjecture of Emil Artin is that an L-function associated to an irreducible representation of a Galois group is entire. Unfortunately, getting analytic information on these L-functions is very difficult. One avenue of approaching the L-functions of Galois representations is proposed by Robert Langlands. He conjectures these L-functions are in a collection of those L-functions coming from automorphic forms associated to another group. The hope then is to use the wealth of tools that are available to study automorphic forms and likewise their L-functions, to get information on the L-functions associated to Galois representations. For example, if one understood the analytic properties of the L-function attached to the 12th symmetric power representation of a GL(2) automorphic form, one could resolve Artin’s conjecture in degree two. We will come back to this example again, as it motivated the origin of the author’s research.

What then are the tools used on the automorphic side? These can be summarized as the converse theorem, the Langlands-Shahidi method, and the trace formula. The first two have had great success, but there seems to be limits in how much information they can produce. For example, the second method uses as its ammunition a finite set of simple Lie algebras. The use of these has been exhausted, to great accolades, by Kim and Shahidi to get the analytic properties of the symmetric fourth power L-function of an automorphic form. As a corollary we have the best known bounds toward the Ramanujan conjectures, which are of great importance in number theory.

The question now is how to get information on higher symmetric power L-functions? A hope is the trace formula. Inspired to study the symmetric 12th power L-function and attack the Artin conjecture, Langlands wrote the paper *Beyond Endoscopy*. In it he proposes to use the trace formula and an extra averaging to get analytic information about the L-function. We will give details of what he does, but first we describe the trace formula, and if we are to go “beyond” endoscopy, what is “endoscopy”?

1. Trace Formula and Endoscopy

The trace formula has many guises—Selberg’s celebrated trace formula, its generalization to higher rank called the Arthur-Selberg trace formula, and Jacquet’s relative trace formula. In spirit, they all express an identity between two different sets of data associated to a group $G$. On one side, called the spectral side, is a sum over all the automorphic representations of the group. The other side, called the geometric side, one has a sum over the conjugacy classes of the group. One can get information about one side by carefully studying the other.
Endoscopy is a certain comparison of automorphic forms on different groups, similar to Langlands’s principle of functoriality. Loosely, representations on the two groups, say $G$ and $G'$, correspond if the attached L-function of the representations are equal. We say the smaller or simpler group $G'$ is endoscopic to $G$ if the representations of the group $G'$ describe the internal structure of the representations of $G$. How is this done? Here is where the trace formula comes into play. If we take the associated trace formulas for the endoscopic groups $G$ and $G'$, they can be compared! Namely, there is a “test” function associated to the trace formula, and if these are chosen correctly for each trace formula, one can get an equality of the two trace formulas. This then gives an equality for the spectral sides. Choosing the correct test functions is a very hard problem known as the Fundamental lemma and only recently solved by Ngô Bao Châu, for which he was awarded the 2010 Fields Medal.

The point is endoscopy does not cover all of the comparisons between groups and their representations, specifically it does not cover the symmetric power representations. Langlands, seeing the limits of endoscopy, then suggested a new approach. Rather than comparing trace formulas on two different groups, one takes a limit and an extra averaging on a single trace formula. Let’s take $G = GL(2)$, and $\pi$ an automorphic form of $G$. Let $\rho$ denote a representation of the dual group of $G$, and let $L(s, \pi, \rho) = \sum_{n=1}^{\infty} \frac{c_n(\pi, \rho)}{n^s}$ denote the associated L-function. One can think of the Dirichlet coefficients $c_n(\pi, \rho)$ as Fourier coefficients associated to the form $\pi$. Then if $g \in C_0^\infty(\mathbb{R}^+), \int_0^\infty g(x)dx = 1$, the sum

$$G_{\pi, \rho}(X) := \frac{1}{X} \sum_n g(n/X)c_n(\pi, \rho),$$

and its size (in terms of $X$), gives data about the associated L-function. If the associated L-function has a simple pole at $s = 1$, then $G_{\pi, \rho}(X) = \text{Res}_{s=1} L(s, \pi, \rho) + O(X^{-\delta}), \delta > 0$. However, it is challenging to get such analytic information about a single automorphic form. Langlands, then incorporated the trace formula into this averaging of the coefficients. One studies,

$$\lim_{X \to \infty} \sum_{\pi} G_{\pi, \rho}(X).$$

Now the trace formula can be used as we have a sum over the spectrum of forms on $GL(2)$, and detect the forms that have a non trivial residue at $s = 1$. A key point being when the L-function has a pole, it is “detecting” a functorial transfer.

2. Examples of Beyond Endoscopy

There is a huge gap between what we know and Langlands suggestion to study the problem for $\rho = \text{sym}^2$. This would investigate

$$\lim_{X \to \infty} \sum_{\pi} \sum_n g(n/X)a_{n^2}(\pi),$$

where $a_n(\pi)$ is the $n$-th Fourier coefficient of $\pi$. I propose to work on bridging this gap by applying beyond endoscopy to the frontier of what is known: $\rho = \text{sym}^3, \text{sym}^4$. This will build on work in my thesis, which I describe below. The first cases of beyond endoscopy are by Sarnak in his letter to Langlands [S] for $\rho=$standard representation. There he showed,

$$\sum_{\pi} \sum_n g(n/X)a_n(\pi) = O(X^{-N}),$$

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for any $N > 0$. This implies the analytic continuation of the standard $L$-function for any cuspidal automorphic form $\pi$. Further, he sketched out an idea of how to study the limit for $\rho = \text{sym}^2$, which was fleshed out and proved in Venkatesh’s thesis [V]. This result can be summarized as the symmetric square $L$-functions of $\pi$ have a pole at $s = 1$ only if $\pi$ is induced from a Hecke character attached to a quadratic field. The analytic difficulties going from the $\text{sym}^2$ to $\text{sym}^3$ representations is considered enormous. We save discussion for $\rho = \text{sym}^3$ for the next section.

In my thesis, one of the cases I study is the Rankin-Selberg $L$-function. To state the theorem, we define the spectral side of the Kuznetsov trace formula. Let

$$K_{n,l}(V) := S_{n,l}(V) + C_{n,l}(V), \text{ where } S_{n,l}(V) := \sum_{\phi} h(V, \lambda_\phi) a_n(\phi) \overline{a_l(\phi)}.$$ 

Here the $\phi$-sum is over discrete automorphic forms on $GL(2)$, and $h(V, \lambda_\phi)$ is a Bessel transform of a test function $V \in C_0^\infty(\mathbb{R}^+)$ included in the sum to ensure convergence. For brevity, we just call $C_{n,l}(V)$ the continuous spectrum contribution. Then the beyond endoscopy calculation in this case takes the form:

**Theorem 2.1.** [H1] Let $l, l'$ be positive integers and $V, W \in C_0^\infty(\mathbb{R}^+)$, then

$$\lim_{X \to \infty} \frac{1}{X} \sum_{n} g(n/X) K_{n,l}(V)K_{n,l'}(W) = \frac{12}{\pi} K_{l,l'}(V \ast W),$$

where $V \ast W$ is a special convolution of $V$ and $W$. $\frac{12}{\pi}$ is the residue of the Rankin-Selberg $L$-function with suitable normalizations.

In order to get deeper information on automorphic forms, one needs to look at these calculations for $L$-functions that are not associated to the standard representation of the dual group. Poles of the standard $L$-function just describe whether the form is cuspidal or not. One $L$-function that gives deeper information is the Asai $L$-function. Let $K = \mathbb{Q}(\sqrt{D})$ denote a real quadratic field with class number one, and $\Pi$ be a nontrivial automorphic representation on $K$. The representation has associated to it Fourier coefficients $\{c_\mu(\Pi)\}$ parametrized by integral ideals $\mu$, and the Asai $L$-function is defined as $L(s, \Pi, \text{Asai}) = \zeta(2s) \sum_{n=1}^{\infty} \frac{c_\mu(\Pi)}{n^s}$. The key aspect of this $L$-function is that if it has a pole at $s = 1$, the form is lifted from an automorphic form on $\mathbb{Q}$. In other words, the Asai $L$-function “detects” base change. Using a beyond endoscopic approach to the Asai $L$-function we prove:

**Theorem 2.2.** [H2] Let $V \in C_0^\infty(\mathbb{R}^+)$. Then for any positive integer $M \geq 0$, and quadratic integer $l, (l, D) = 1$,

1. \{Cuspidal contribution\}

$$\frac{1}{X} \sum_{n, x \in \mathbb{N}} g(m^2 n/X) \sum_{\Pi \neq 1} h(V, \nu_\Pi) c_n(\Pi) c_l(\overline{\Pi}) =$$

$$2\pi(1 + \frac{1}{D}) \left( \sum_{r \in \mathbb{N}} \sum_{\phi_{l,D} \neq \phi_{\mu}} h(V, t_\phi) a_{\mu'}(\phi_{l,D}) a_1(\overline{\phi_{l,D}}) + \sum_{\phi_{k,D}} h(V, k_\phi) a_{\mu'}(\phi_k) a_1(\overline{\phi_k}) \right) + O(X^{-M}).$$

2. \{Coefficient comparison\}

If $\Pi = BC_{K/\mathbb{Q}}(\phi)$, then $c_l(\Pi) = \sum_{r \in \mathbb{N}} a_{\mu'}(\phi)$.
The sum on the RHS is of automorphic forms $\phi_{j,D}$ of level $D$ with nebentypus $\chi_D$ the quadratic character, and either $j$ equals $t$ for eigenvalue $1/4 + t^2$ or minimal weight $k$, depending on the representation the form comes from. The theta forms constructed by Maass, denoted $\theta_{\omega \mu}$, come from Hecke characters over the field $K$; these are removed as they base change to Eisenstein series.

The extra summation in $\pi$ is a new feature to the beyond endoscopic approach which gives not only information around $s = 1$, but analytic continuation to the whole plane. We are currently working on adding the extra summation for the Rankin-Selberg $L$-function, [H4]. This provides strong evidence that the trace formula can be used to get all the analytic information one gets from the converse theorems and the Langlands-Shahidi method. Perhaps even more advantageous is that one gets the information all at once and not a form by form case.

### 3. Current/Future Projects

My research goals involve 4 projects, each of which relate to the beyond endoscopy calculation.

**A.) Poles of the Symmetric Cube $L$-function**

We now come back to the symmetric cube $L$-function. We consider the problem for forms of full level. Here we expect the symmetric third $L$-function to not have any poles, and therefore get

$$\lim_{X \to \infty} \frac{1}{X} \sum_{n \in \mathbb{N}} g(n/X) \left( \sum_{\phi} h(V, \lambda_{\phi}) a_{\pi^3}(\phi) \bar{a}_{\pi^3}(\phi) + \{\text{cont.spectrum}\} \right) = 0.$$ 

Without the limit, the LHS can be rewritten after a use of the Kuznetsov trace formula and Poisson summation as

$$\frac{1}{X^{3/2}} \sum_c \frac{1}{c} \sum_{x(c)^*} e\left(\frac{xt}{c}\right) \sum_m \sum_{k(c)} e\left(\frac{xk^3 - mk}{c}\right) W_m\left(\frac{c}{X^{3/2}}\right),$$

where $W_m(y) = \int_{-\infty}^{\infty} e\left(\frac{-tm}{\sqrt{y}X}\right) g(t)V(4\pi \sqrt{t}) dt$. So we hope to get the sum to be of size $O(X^{3/2-\delta})$, $\delta > 0$. Notice the $m$-sum is of size $O(\sqrt{X})$, by the smoothness and compactness of $V$, so one hopes the sum in $c$ is $O_m(X)$. Evidence of this comes in the case $m = 0$.

**Theorem 3.1.** [H3] 

$$\frac{1}{X^{3/2}} \sum_c \frac{1}{c} \sum_{x(c)^*} e\left(\frac{xt}{c}\right) \sum_{k(c)} e\left(\frac{xk^3}{c}\right) W_0\left(\frac{c}{X^{3/2}}\right) = O(X^{-1/2}).$$

This certainly beats the trivial bound of $O(X^{3/2})$, and even if one allows square root cancellation in the $k$ and $x$-sums, one still gets $O(1)$.

If instead one looks at the problem over a field adjoined the cube roots of unity, we can do much better! If here we look at automorphic forms with central character the cubic residue character ($z_3$) then one can get more,

**Theorem 3.2.** [H3] 

$$\frac{1}{X^{3/2}} \sum_c \frac{1}{c} \sum_{x(c)^*} e\left(\frac{xt}{c}\right) \sum_{k(c)} e\left(\frac{xk^3 + mk}{c}\right) W_m\left(\frac{c}{X^{3/2}}\right) = O(X^{1/4}).$$
Note an application of the Weil bound on the Kloosterman sum in this setting would give only $O(X^\frac{3}{4})$. This certainly can be improved with a more detailed investigation of metaplectic forms.

These two theorems rely on the identity of Duke and Iwaniec in [DI],

$$\sum_{k(c)} e\left(\frac{xk^3 + mk}{c}\right) = \sum_{y(c)*} \left(\frac{xy}{c}\right) e\left(y - \frac{m^3 y}{3c}\right),$$

where $(xm, c) = 1$. It compares cubic exponential sums with Kloosterman sums twisted by a cubic character. After applying the identity, one can then use a Dirichlet series of Patterson [P1], with coefficients being squares of Gauss sums over the field $\mathbb{Q}(\omega)$, where $\omega$ is a cube root of unity. One goal to work on would be to get this identity or something similar to hold for primes $p \equiv 2(3)$, so as to get a complete picture of the poles of the symmetric cube $L$-function. Another aspect of this project would be to do the computation for forms of prime level $N \equiv 1(3)$, with a character of cubic order times a quadratic character of discriminant $N$. These forms should give symmetric third power $L$-functions having poles, or a nontrivial contribution to (3.1).

B.) Fundamental Lemma and Beyond Endoscopy

Mao and Rallis [MR] state that the identity (3.2) is key to solving the Fundamental Lemma in the relative trace formula comparing automorphic forms from the three-fold cover of $\tilde{SL}_2$ to $SL_2$ itself. A very similar identity for $\sum_{k(c)} e\left(\frac{k^4 + kc}{c}\right)$, is proved in [Y1]. Such an identity seems natural to a beyond endoscopy attack on the symmetric fourth power $L$-function, and hopefully be understandable as a consequence of the Fundamental Lemma as well. The analogue of the Dirichlet series used for the symmetric third calculation, would hopefully come from a similar residual Eisenstein series as in [KP]. We mention also in the proof of quadratic base change in [Y2], another analogous exponential identity is needed for the Fundamental Lemma there. This same identity is also vital in the beyond endoscopy approach of a similar problem in proving Theorem 2.2, [H2].

Other than the examples just mentioned, there is not a clear picture of how the Fundamental Lemma comes into play for the beyond endoscopy approach. An important goal in the next year is to answer, is their a natural way to see these exponential identities needed for studying the symmetric power $L$-functions from Ngô’s resolution of the Fundamental lemma?

C.) Exponential sum technology.

It is easy to notice that in the geometric side of the trace formula one encounters sums of exponential sums. These in themselves are of great interest. One conjecture of Patterson [P2] is for $f$ a polynomial of degree $n$ with integral coefficients

$$\sum_{c \leq X} \sum_{x(c)} e\left(f\left(\frac{x}{c}\right)\right) \sim KX^{1+\frac{1}{n}},$$

for some constant $K$. A resolution of this would be of great help to the circle method and in diophantine equations. The problem was resolved in [LP] for $n = 3$ over the field $\mathbb{Q}(\omega)$, where $\omega$ is a cube root of unity. The techniques used included automorphic forms, specifically a technique of Sarnak and Goldfeld [GS]. The same unsolved problem over $\mathbb{Q}$ shows up in the analysis of the symmetric cube $L$-function above, it is worth investigating if such problems can be resolved with the beyond endoscopy approach.
This goal would also be a great project to get undergraduates involved in. An eager undergraduate could study these sums with math software such as Sage, Magma, or Maple.

D.) Arithmetic functions on sparse sequences.

One can see from above, certain beyond endoscopy questions study arithmetic functions on “thin sets.” For example, averages of Fourier coefficients at the $n^k$-powers. One wants to get power savings or cancellation in such sums. Let $\phi$ be a holomorphic form of weight 2 and $g(x) = x^2 + d$, a non-split polynomial, then an example of Templier [T] is,

$$\sum_{X < n < 2X} a_{n^2 + d}(\phi) = O(\phi(X^{1-\delta})).$$

Such results, beautiful to the applicant, have applications to proving subconvexity for and studying moments of $L$-functions. In the next years, an important extension of such questions would be to look at the same sum restricted to primes. Obviously, this is much harder, but the incorporation of the trace formula,

$$\sum_{\phi} \sum_{p \leq X} a_{p^2+1}(\phi),$$

or extra averaging over the level or nebentypus should help in making the problem tractable. This would be an analogue of a conjecture of Friedlander and Iwaniec [FI], that for the divisor function $\tau(n)$,

$$\sum_{p \leq X} \tau(p^2 + 1) \sim cX.$$

The advantage for automorphic forms is one has a richer set of tools to use then for the classical problem. These problems will also require sieving methods which we would like to use more in automorphic forms.

References


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