Workshop on CALIBRATIONS
June 26, 2006 - June 30, 2006

Organizers: Robert Bryant, Xiaobo Liu, and Pit-Mann Wong

The theory of calibrations finds its genesis in a 1982 *Acta Mathematica* paper of Reese Harvey and Blaine Lawson. Both Harvey and Lawson attended the AIM workshop and both spoke on the morning of the second day. The spirit of calibrated geometry is to understand the structure of a manifold by studying a distinguished family of submanifolds. This approach leads to insights that are unavailable from the more classical points of view using coordinate charts or tensor fields.

Let X be a Riemannian manifold and φ a closed, exterior p-form satisfying

$$\varphi|_\xi \leq \mathrm{vol}_\xi$$

for all oriented tangent p-planes ξ on X. We call φ a *calibration* on the manifold X. A p-dimensional submanifold $M \subset X$ is calibrated by φ if $\varphi|_M$ is the volume form of M. Calibrated geometry studies distinguished submanifolds calibrated by some calibrations. The fundamental theorem of calibrated geometry says this:

> **Theorem:** Let φ be a calibration on X. Then every closed submanifold $M \subset X$ that is calibrated by φ is volume minimizing in its homology class.

Of particular interest, at least at the beginning, are forms that are parallel. Examples of such calibrations are forms with constant coefficients in Euclidean spaces. In this case, at least locally, the problem of finding calibrated submanifolds is reduced to solving a system of partial differential equations. These equations are, in a natural sense, generalizations of the classical Cauchy-Riemann equations.

The theory of calibrations has grown to be important because of its many applications to gauge theory and mirror symmetry. We describe just one example. The fundamental idea of mirror symmetry is that there is a correspondence between the symplectic geometry of a Calabi-Yau manifold and the complex geometry of its mirror partner. A significant problem in mirror symmetry is the so-called SYZ conjecture of Strominger, Yau, and Zaskow. The question is whether the mirror of a Calabi-Yau manifold X can be obtained by dualizing the fibers of a special Lagrangian toric fibration of X.

A number of important topics were discussed during the work shop. Many of these are connected with the concept of plurisubharmonicity and its generalizations.[1] The main ones are listed below:

(a) Calibrated cycles and mirror symmetry;

(b) Calibrated geometry and gauge theory;

(c) Calibrated plurisubharmonicity;

(d) Integrable systems;

(e) Construction of submanifolds calibrated by special Lagrangian forms, associative and coassociative forms;

(f) Spinors and calibrations;

(g) Exterior differential systems and calibrations;

The AIM workshop on calibrations started with a survey by G. Tian on calibrated geometry and gauge theory followed by D. Joyce's lecture on problems in special Lagrangian geometry and their applications to mirror symmetry. The list of problems can be found in Joyce's notes posted on the workshop webpage. Among these are problems arising from Kontsevich's homological mirror symmetry conjecture:

For a pair of mirror Calabi-Yau manifolds $(M, \widetilde{M})$, the bounded derived category of the Fukaya category $D^b(F(M))$ (which is built from the theory of pseudo-holomorphic discs and intersections of Lagrangian submanifolds in M) is equivalent to the bounded derived category of coherent sheaves $D^b(\text{coh}(\widetilde{M}))$ on the mirror $\widetilde{M}$. This category-theoretic interpretation gave rise to considerable discussion in the first few days of the workshop.

The second day of the workshop was devoted to the recent work of B. Lawson and R. Harvey on calibrated (or ϕ)-plurisubharmonic functions on

[1] Classically, a function on a domain in $\mathbb{C}^n$ is *plurisubharmonic* if it is subharmonic along each complex line. Plurisubharmonic functions arise naturally as the log-moduli of holomorphic functions.

a non-compact calibrated manifold (M, ϕ). This is motivated in part by the theory of Stein manifolds[2] in complex analysis and the desire to develop the function-analytical aspect of calibration. The highlight was that the concept of (strictly) plurisubharmonic exhaustion in Stein theory has a good calibrated analogue (and the complex case is simply the special case where M is complex and ϕ is a Kähler form). Moreover, it was shown that the Sibony-Duval potential-theoretic results in complex analysis carry over, to a great extent, to ϕ-plurisubharmonic functions. For compact manifolds equipped with a calibration, analogues of the complex Hodge conjecture can be formulated. These are new and exciting directions in calibration theory that are to be explored further. The participants are encouraged to consult the notes of the talks of Harvey and Lawson that are posted on the workshop web site.

There were problem sessions on both days and a number of open problems were raised, discussed and refined; a complete list will be made available shortly.

A number of issues of great interest to many of the participants were identified during a planning session. Two of the issues: integrable systems and mirror symmetries were chosen to be discussed on the third day. The mirror symmetry discussion started by a lecture on "String theory and calibrations" by Melnikov. Among other things, the motivation for the SYZ conjecture from a physicist's point of view was presented (via D-branes wrapping around cycles). The mathematical discussion on mirror symmetry focused mainly on understanding more precisely the conjectural equivalence of the Fukaya category and the derived category of coherent sheaves (presented by Joyce and Ruan).

The discussion on integrable systems started with a presentation by McIntosh on how spectral curves arise in the solution of integrable systems written in Lax pairs. The T^2-cones in $\mathbb{C}^3$ that are special Lagrangian have been extensively studied (and, one could say, classified by spectral data). The problem of doing the same for associative T^2-cones in $\mathbb{R}^7$ is still open. In particular, the case whose link in S^6 is a non-super dual J-holomorphic curve corresponds to a solution of the G_2 Toda equations. The spectral data for minimal tori in S^6 is quite well understood, but detecting the extra condi-

[2] A Stein manifold is a generalization of the idea of domain of holomorphy. Roughly speaking, a Stein manifold is a complex manifold that has plenty of global holomorphic functions, and is holomorphically complete. An important characterization of Stein manifolds is that such a manifold has a plurisubharmonic exhaustion function.

tion required for the G_2 conditions is more subtle. A key is to understand how the spectral data provides a frame for the primitive harmonic map[3] into $G_2/T^2 \subset SO(7)/T^3$. This is one of the main problems that I. McIntosh, E. Carberry, and E. Wang spent a lot of time on during the workshop. After the lecture, a group gathered to discuss some ideas on the role of integrable systems in understanding associative, coassociative, and Cayley submanifolds in Euclidean spaces of dimensions 7 and 8 respectively.

The discussion continued during lunch and lasted all afternoon, during which the mirror symmetry group and the integrable system group split off into different parallel discussion sessions. The mirror symmetry group attempted to understand the precise mathematical formulation of the language and the intuition of the physicists in dealing with mirror symmetry.

The fourth day of the workshop began with a discussion session on problems arising from the work of Harvey and Lawson, particularly on the construction of many ϕ-plurisubharmonic functions. There are many problems in this direction that are probably quite accessible with the current techniques. The following problem was discussed: Characterize those calibrations $(\mathbf{C}^n, \phi)$ for which local convexity implies global convexity. This is known to be true for the Kähler calibration (this is the celebrated Levi-problem in complex analysis resolved, by H. Grauert, via the solution of the $\bar{\partial}$-equation). On the other hand, there exist calibrations for which this is false. All known counter-examples are non-elliptic (that is, the form ϕ does not contain all variables). Thus, as a first step one may ask if ellipticity is sufficient for the solution of the calibrated-Levi problem

The last day of the workshop started with a lecture by C. Leung on extending mirror symmetries to manifolds with $G2$ or Spin(7) holonomy, followed by a lecture by D. Fox on constructing coassociative submanifolds. In the afternoon, R. Bryant talked about exterior differential systems and applications to calibrations. The workshop was successful in many respects. It was a considerable learning experience for those who attended, and several new collaborations have grown from the workshop.

[3]A harmonic map between two Riemannian manifolds is one that minimizes the energy functional. In the case that the domain of the map is one-dimensional, the harmonic map is a generalization of the concept of geodesic. In the case that the range of the map is Euclidean, the harmonic map is a generalization of the concept of harmonic function.