Workshop on Classification Theory for Abstract Elementary Classes
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Organizers: Rami Grossberg and Monica VanDieren

Modern mathematical logic was born about one hundred years ago. It was the product of concerns about the coherence of mathematics, and about the way that it was recorded and communicated. Today logic has blossomed into an independent mathematical field with many fascinating problems of intrinsic interest and merit.

The workshop on the theory of abstract elementary classes was a far-reaching event, which was particularly exciting because of the many applications of this abstract set of ideas to concrete areas of mathematics. Abstract elementary classes provide a mechanism for studying model theoretic properties (such as an abstract notion of independence) of classes of mathematical structures which need not be axiomatizable by a first-order logic. Some early results of Shelah on abstract elementary classes provided a solution strategy for the Main Gap Theorem, the cumulative result in Shelah's first order classification theory book. But what perhaps drives interest in the field today and inspiration for this workshop are recent breakthroughs of a test conjecture made by Shelah in 1978 and applications of partial solutions of this conjecture to number theory.

One of the first dramatic applications of the theory of abstract elementary classes was to the celebrated Schanuel conjecture in number theory. The topic here is transcendence theory. A number is *algebraic* if it is the solution of a polynomial equation with integer coefficients. For example, $\sqrt{2}$ is algebraic because it is the solution of the polynomial equation $x^2 - 2 = 0$. A number is transcendental if it is not algebraic. It is quite difficult to prove that any particular number is transcendental. But it is known, for example, that the numbers π and e are transcendental (the proofs of both these statements are quite recondite). It was a startling result of Georg Cantor in the late nineteenth century that "most" real numbers are transcendental. He proved this theorem by abstract methods, without actually identifying any particular transcendental number.

It is of great interest to be able to identify and analyze transcendental numbers. This is important not just for the sake of abstract mathematics but also because such results are useful in coding theory and cryptography. The algebraic numbers are a bit easier to manipulate than the transcendental numbers because they are closed under addition and multiplication. Thus it must be that either $\pi + e$ or $\pi - e$ is transcendental. Because if they were both algebraic then their sum would be algebraic hence π would be algebraic—which it is not. It can now be shown that Schanuel's conjecture implies that $\pi + e$ is transcendental.

Boris Zilber, a participant of the workshop, has a number of striking new results in this field. Zilber considered the following example of an abstract elementary class: the collection of all algebraically closed fields of characteristic zero with an exponential operation that satisfies Shanuel's conjecture. Zilber showed that this class contained an infinite field in this class which may not be large enough to be the complex numbers. Zilber was able to show that fields of this particular size in the class were unique up to isomorphism. Appealing to model theoretic techniques reminiscent of early work of Shelah in the classification theory of abstract elementary classes, Zilber used this unique field and certain properties of independence relations on this field to build up a field of the same cardinality as the complex numbers in this class. Further, Zilber found that there was exactly one field of this cardinality in the class, up to isomorphism. A classic result of algebra tells us that this field must be the complex numbers. From the axiomatic point of view, Zilber showed that there was a function on the complex numbers very much like e^x for which Schanuel's conjecture is true.

One of the key ideas in Zilber's work was the uniqueness result. This result is a particular case of Shelah's Categoricity Conjecture. Shelah posed this conjecture to guide the development of a classification of abstract elementary classes in a similar role that Los' conjecture had in the development of first order model theory in the late sixties and early seventies. Shelah's conjecture states that if an abstract elementary class has a unique model (up to isomorphism) of a particular cardinality, then there will be unique models (up to isomorphism) of other cardinalities as well. This would be a powerful result in creating models and examples, as Zilber's investigations have shown.

The workshop was attended by many graduate students - especially from U.C. Berkeley. This subject area is particularly fruitful for potential thesis problems, and the organizers made special efforts to bring the students up to speed so that they could participate in all the workshop activities. The AIM workshop developed and explored the theoretical aspects of abstract elementary classes and identified many new connections and applications of model theory to various classes of fields and groups.