Numerical Invariants of Singularities and Higher-Dimensional Algebraic Varieties
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It is a matter of great interest to understand the singularities of complex polynomials or analytic functions of many variables. One wishes to attach a numberical invariant to the singularity that somehow measures its "severity".

The situation is transparent for functions of one variable. Suppose that a polynomial or analytic function $f(x)$ vanishes at (say) the origin. Then we all learn as undergraduates how to measure its order of vanshing there. Namely we write f in the form

$$f(x) = a_m x^m + \text{(terms of higher order)}$$

with $a_m \neq 0$. In this circumstance we say that f has a *zero of order* m at 0, and this integer m completely determines the local behavior of f near 0.

The situation gets much more interesting for polynomials or analytic functions of two or more variables. Consider for simplicity a function $f(x, y)$ of two variables, and suppose that f vanishes at the origin, i.e. $f(0, 0) = 0$. As in the one-variable case, we can define the *order of vanishing* of f at $(0, 0)$ to be the least degree of a monomial appearing in the Taylor series of $f(x, y)$. This gives a good first indication of the local behavior of f, but it by no means gives the whole story. For example, both polynomials

$$g(x, y) \;=\; x^2 - y^2 \quad , \quad h(x, y) \;=\; x^3 - y^2$$

vanish to order 2 at the origin, but the second is certainly more complicated than the first: if we plot the zero loci of each, we find that g cuts out two lines crossing transversely, whereas h defines a cusp.

Starting in the 1980s with work of Varchenko, a more delicate invariant emerged, involving considerations of integrability. Going back to the one-variable situation for the moment, we can compute the order of vanishing of $f(x)$ by asking for which exponents c the integral:

$$\int \frac{1}{|f(x)|^{2c}}\, dx$$

is finite in a neighborhood of 0: if $\mathrm{ord}_0(f) = m$, then the supremum of all such μ is exactly $\frac{1}{m}$. The same procedure works for functions f several variables, and defines an invariant $c_0(f)$ known variously as the *complex singularity exponent* or *log-canonical threshold* of f.

The intuition is that smaller values of $c_0(f)$ reflect more dramatic singularities at 0. For instance, going back to our two functions $g(x, y)$ and $h(x, y)$, one finds that

$$c_0(g) = 1 \quad , \quad c_0(h) = \frac{5}{6},$$

so in fact this invariant sees that h is more complicated than g.

The complex singularity exponent emerged independently in several branches of mathematics during the past twenty years, but it is only fairly recently that it has achieved real prominence. One of the main goals of the workshop was to learn about and study the appearance of this and related invariants in surprisingly many places. To give a taste of one of these, suppose that $f(x, y) \in \mathbf{Z}[x, y]$ has integer coefficients, and fix a prime number p. Then one can look at solutions of the congruence

$$f(x, y) \equiv 0 \pmod{p^m},$$

and count the number N_m of such as m varies. Igusa proved in the 1970s (before $c_0(f)$ had been defined!) that under some mild technical hypotheses $c_0(f)$ governs the asymptotics of N_m:

$$N_m \approx p^{m(2-c_0(f))}.$$

Analogues of $c_0(f)$ have also started to come up in positive characteristic commutative algebra and real analysis. Several instructional sessions at the workshop were devoted to explicitly working out examples in these different approaches.

One of the reasons for the recent interest in these and related invariants is that they arise in the classification theory of higher dimensional algebaic varieties, where measures of singularities play a prominent role. One exciting element of this workshop is that some of the participants reported on substantial recent progress in this area. Around 1990, Shigefumi Mori proved in dimension three that the “canonical ring” of a variety is finitely generated; he was awarded the Fields medal for this work. After important contributions by Shokurov, groups spearheaded by Hacon/McKearnan and Yum-Tong Siu have made considerable progress on generalizing Mori’s result to all dimensions. It now seems likely that one will soon have in hand a classification result for all varieties that generalizes the classical picture of grouping Riemann surfaces according to whether they have positive, zero or negative curvature.

In summary, the idea of singularities is one of the oldest in algebraic geometry and related areas. But recent research has opened many new doors and connections in this area, and the workshop attempted to explore and encourage these.