Workshop on GENERALIZED KOSTKA POLYNOMIALS

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One of the fundamental ideas of modern algebra is that of the "symmetric polynomial". A polynomial is *symmetric* if is invariant under permutations of the variables. As an instance, the polynomial $x_1x_2 + x_2x_3 + x_1x_3$ is a symmetric polynomial of three variables. We wish to consider the ring $\Lambda = \Lambda[x_1, x_2, \ldots, x_n]$ of symmetric polynomials in infinitely many variables.[1]

There are many different bases for the ring Λ. Perhaps the most important of these is the Schur polynomials. We are interested in the relationship of Schur polynomials to partitions. For us a *partition* of the positive integer n is a weakly decreasing sequence of integers which sums to n. To understand Schur polynomials, fix such an integer n and consider a partition. We illustrate with Figure 1 a partition (commonly termed a *Ferrer's diagram*) of the number 9:

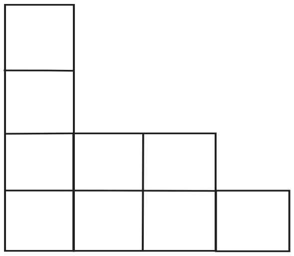

Figure 1

A standard algebraic notation for this particular partition would be $(4, 3, 1, 1)$.

We have a set of rules by which we place elements of our alphabet into the boxes of the Ferrer's diagram. We use the term *tableau* to denote a filling of a Ferrer diagram. Suppose that our alphabet consists of the positive integers. Then the rules for placing members of the alphabet into the boxes are

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[1] The "infinitely many variables" is a linguistic convenience so that we do not have to specify in advance how many variables we will consider.

(a) The rows should be nondecreasing from left to right.

(b) The columns should be strictly increasing from bottom to top.

A diagram satisfying **(a)** and **(b)** is called a Young tableau. Schur polynomials are interesting because they correspond to the Young tableaux.

In case $n = 2$ then the tableau could be the simple configuration in Figure 2:

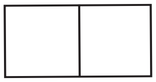

Figure 2

Possible ways to fill in letters of our alphabet $\{1, 2, 3\}$, according to the rules, are illustrated in Figure 3:

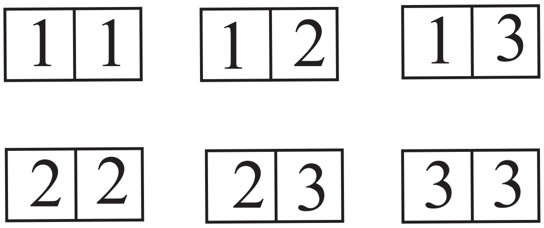

Figure 3

Now we associate to teach of these labeled tableaux a monomial. The first tableau has two 1's, so the associated monomial is x_1^2. The second tableau has a 1 and a 2, so the associated monomial is x_1x_2. Continuing in this manner, we generate a total of six monomials. Their sum,

$$x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3\,,$$

is certainly a symmetric polynomial. If we let λ denote the partition then this particular function, called a *Schur function*, is denoted s_λ.

A perhaps more enlightening example is given by the tableau in Figure 4.

The possible labelings, taken from the alphabet $\{1, 2, 3\}$, are shown in figure 5.

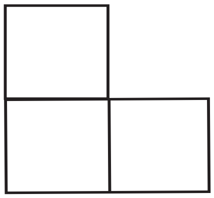

Figure 4

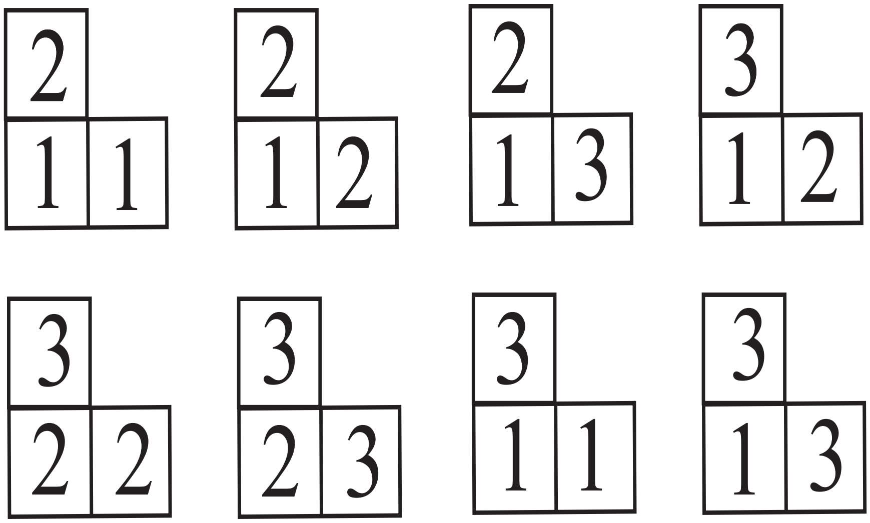

Figure 5

We add together the resulting monomials to obtain this symmetric function:

$$x_1^2x_2 + x_1x_2^2 + 2x_1x_2x_3 + x_2^2x_3 + x_2x_3^2 + x_1^2x_3 + x_1x_3^2\,.$$

The polynomials that we produce in this way form a basis for the symmetric polynomials.

Notice that, in each of our examples, the degree of the resulting Schur polynomial s_λ is the number of boxes.

It is convenient to represent our construction symbolically by

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in SSYT(\lambda)} x^T\,.$$

A bit of explanation is required here. The acronym "SSYT" stands for "semi-standard Young tableau". These are the entries from one of our tableaux, as in Figures 3 or 5. Of course x^T is multi-index notation representing the product of those x_i raised to the power given by the number of instances of i that appear in T. [By contrast the *standard* Young tableaux are those in which each of the integers from 1 to n appears once and only once.

Of course the symmetric polynomials form a ring. It is worth concentrating explicitly on the product operation. If s_λ and s_μ are two of our Schur polynomials, then we may write

$$s_\lambda(x) \cdot s_\mu(x) = \sum_\nu c^\nu_{\lambda\mu} s_\nu(x)\,.$$

What is going on here is that the product of s_λ and s_μ is another symmetric function, so certainly can be written as a linear combination of other Schur polynomials (because they form a basis). It is matter of great interest to understand the coefficients $c^\nu_{\lambda\mu}$. These are called the Littlewood-Richardson coefficients, and it turns out that they are nonnegative integers (this assertion is quite difficult to prove).

The Kostka polynomials group at this workshop wants to generalize the symmetric function theory that we have described above. They have a theory for multiplying tableaux that is in fact non-commutative. It is possible to study the *commutative* algebraic structure of symmetric polynomials by imbedding it in the *non-commutative* tableaux theory.

Key to this study is the so-called Robinson-Schenstead-Knuth (RSK) bijection

$$S_n \leftrightarrow \bigcup_\lambda \Big[SYT(\lambda) \times SYT(\lambda) \Big]\,.$$

This associates a permutation with a pair (P, Q) of tableaux with the same shape. The workshop participants wish to lift this construct to the affine world (in which we model our analyses on all the integers $\mathbb{Z}$ rather than just the positive integers $\mathbb{N}$. They have an analog of the Schur function connected to the affine symmetric group.

We provide now an example of the bijection discussed above:

Schur polynomials have applications to algebraic geometry, particularly to certain cohomology rings. They arise also in representation theory, combinatorics, and mathematical physics. For the latter, the symmetric polynomials reflect the coefficients of the operator expansions that arise in certain phase transition formulas of conformal field theory.

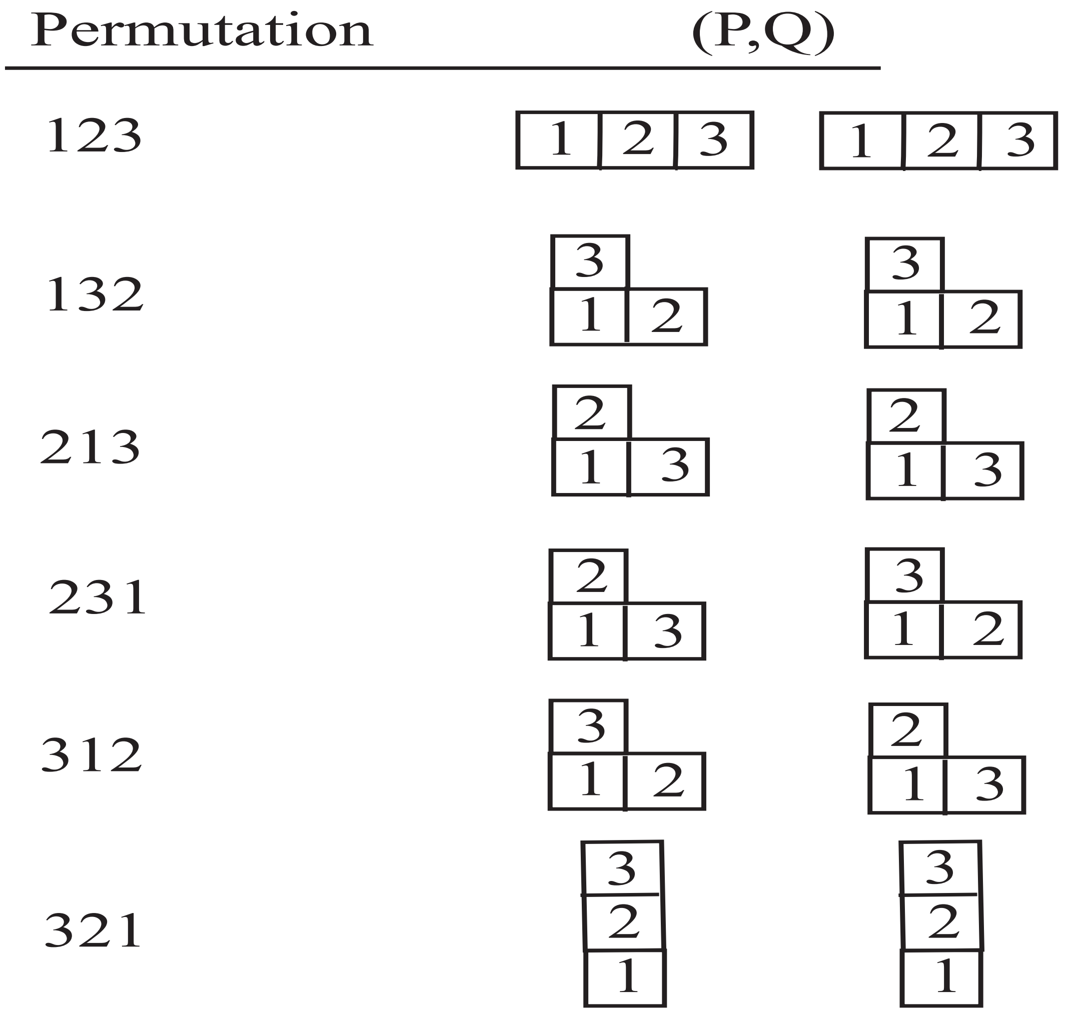
Permutation
(P,Q)
123
1 2 3
1 2 3
132
3
1 2
3
1 2
213
2
1 3
2
1 3
231
2
1 3
3
1 2
312
3
1 2
2
1 3
321
3
2
1
3
2
1

Figure 5